COLLECTIVE CHOICE RULES: A CLASSIFICATION USING THE OWA OPERATORS

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Summary

We propose a general scheme of collective choice rule that covers a number of well-known rules. Our point of departure is, first, the set of fuzzy preference relations, and second, the linguistic aggregation rule proposed by Kacprzyk [2-4]. We reconsider this rule on a more abstract level and use the OWA operators instead of Zadeh's fuzzy linguistic quantifiers. All collective choice rules from our general scheme are applicable either to classical or fuzzy preferences.

1. INTRODUCTION

In real life decision situations often the aggregation of preferences is needed. A lot of rules governing such an aggregation have been devised. In particular, rules producing a set of decisions (options, alternatives etc.) that may be rationally treated as favored by all individual preferences being aggregated are known as collective choice rules.

In this paper we propose a general scheme of collective choice rule that covers a number of well-known rules. Our point of departure is the linguistic aggregation rule as proposed by Kacprzyk [2-4]. We re-consider this rule on a more abstract level and use OWA operators instead of originally employed linguistic quantifiers in the sense of Zadeh. We obtain mentioned earlier general scheme. Our approach is quite general. We consider individual fuzzy preference relations as a point of departure. Thus, all particular collective choice rules recovered from our general scheme are applicable either to classical or fuzzy preferences.

We introduce the notation used and remind the concept of fuzzy preference relation. Second, we briefly discuss the concept of linguistic quantifiers and fuzzy majority to be used in a specific, linguistic collective choice rule. Finally, we propose general scheme of collective choice rule and discuss some its special cases.

2. FUZZY PREFERENCE RELATIONS

In order to study the possibly general case we assume the preferences to be aggregated are represented as fuzzy preference relations. Fuzzy relations allow for a more natural expression of the individual preferences, which often are not clear-cut. In what follows the following notation will be used. Let $S = \{s_1, ..., s_M\}$ be a finite set of options and $X = \{x_1, ..., x_N\}$ the set of individuals. Fuzzy preference relation R is a fuzzy subset of the $S \times S$ space. It may be identified with its membership function:

$$\mu_R(s_i, s_j) = \begin{cases} 1 & \text{definite preference} \\ c \in (0.5,1) & \text{preference to some extent} \\ 0.5 & \text{indifference} \\ d \in (0,0.5) & \text{preference to some extent} \\ 0 & \text{definite preference} \end{cases}$$

The degree of preference is here interpreted in a continuous manner, i.e., when the value of a membership function changes from the one slightly below 0.5 to the one slightly above 0.5, there is no abrupt change of its meaning - both values more or less correspond to the indifference. In the other words, the particular values of a membership function $\mu_R(s_i, s_j)$ express uncertainty as to the actual preferences, highest in the case of 0.5 and lowest in the case of 1.0 and 0.0.

The particular values of a membership function $\mu_R(s_i, s_j)$ may be interpreted in a different way. For example, Nurmi [5] assumes, that $\mu_R(s_i, s_j) > 0.5$ means definite preference of s_i to s_j and the particular values from the (0.5,1] interval express the intensity of this preference. In what follows, we refer to this interpretation as Nurmi's interpretation.

Usually, the fuzzy preference relation is assumed to meet certain conditions, most often reciprocity and transitivity. For our main result, none of these properties are directly relevant. Nevertheless, for some parts of our presentation it is suitable to assume the reciprocity property, i.e., we will assume the following condition holds:

$$\mu_R(s_i, s_j) + \mu_R(s_j, s_i) = 1, \forall i \neq j$$

Such relations are known as fuzzy tournaments. Assuming reasonable (small) cardinality of the set S, it is convenient to represent a preference relation R_k of the individual k in the form of a matrix:

$$[r_{ij}^k] = [\mu_{R_k}(s_i, s_j)], \forall i, j, k \,.$$

3. LINGUISTIC QUANTIFIERS AND OWA OPERATORS

Fuzzy majority constitutes a natural generalization of the majority concept for the case of a fuzzy domain. It may be directly related to the linguistic quantifiers, which often appear in a natural language discourse. Linguistic quantifiers exemplified by expressions like "most", "almost all" etc. allow for a more flexible quantification of entities than the classical general and existential quantifiers. There exist a few approaches to the linguistic quantifiers modeling.

Basically, we are looking for the truth of a proposition of the following type:

"Most objects posses certain property"
what may be formally expressed as follows:

$$\underset{x \in X}{\mathrm{Q}} \; P(x) \qquad (1)$$

where Q denotes a fuzzy linguistic quantifier (in this case "most"), $X = \{x_1, \ldots x_m\}$ is a set of objects, $P(\cdot)$ corresponds to the property. It is assumed that the property P is fuzzy and its interpretation may be informally equated with a fuzzy set and its membership function, i.e.:

$$\mathrm{truth}(P(x_i)) = \mu_P(x_i)$$

One well-known approach was proposed by Zadeh [11,12], is called the calculus of linguistically quantified propositions. Here, a linguistic quantifier is represented as a fuzzy set $Q \in \mathrm{F}([0,1])$, where F(A) denotes the family of all fuzzy sets defined on A. For our purposes and for some practical reasons, its membership function should be assumed piece-wise linear. Thus the fuzzy set corresponding to the fuzzy quantifier Q ("most") may be defined by, e.g., the following membership function:

$$\mu_Q(y) = \begin{cases} 1 & \text{for } \; y \geq 0.8 \\ 2y - 0.6 & \text{for } \; 0.3 < y < 0.8 \\ 0 & \text{for } \; y \leq 0.3 \end{cases} \qquad (2)$$

In order to determine the truth of the proposition (1) the following formula is employed:

$$\mathrm{truth}(\mathrm{Q}P(X)) = \mu_Q\left(\sum_{i=1}^{N} \mu_P(x_i)/m\right) \qquad (3)$$

where m=card(X).

Another approach to linguistic quantifiers modeling consists in the use of Yager's OWA operators [9-10]. OWA operator O of dimension n may be briefly described as follows:

$$O : \Re^n \to \Re$$

$$W = [w_1, \ldots, w_n], \quad w_i \in [0,1], \quad \sum_{i=1}^{n} w_i = 1$$

$$O(a_1, \ldots a_n) = \sum_{j=1}^{n} w_j b_j, \quad b_j \text{ is } j\text{-th largest of the } a_i$$

Thus, an OWA operator is fully defined by a weight vector W. Some correspondence between OWA operators and linguistic quantifiers in sense of Zadeh is established by the well-known formula:

$$w_i = \mu_Q(i/n) - \mu_Q((i-1)/n) \qquad (4)$$

More precisely, using this formula we may define an OWA operator that behaves similarly to a Zadeh's linguistic quantifier given by the membership function μ_Q.

The OWA operators provides us with a better representation of classical quantifier:

$\forall \to W = [0, \ldots, 0, 1]$ $\qquad O_\forall$
$\exists \to W = [1, 0, \ldots, 0]$ $\qquad O_\exists$

The following weight vectors define other OWA operators useful for our purposes:

classical crisp majority $\qquad O_{maj}$

$$W = [0, \ldots, 0, 1, 0, \ldots, 0] \quad w_{(n/2)+1} \text{ or } w_{(n+1)/2} = 1$$

average $\qquad O_{avg}$

$$W = [1/n, \ldots, 1/n]$$

most $\qquad O_{most}$

(the weight vector may be, e.g., calculated using (1) and (4)).

4. FUZZY MAJORITY IN COLLECTIVE CHOICE

Collective choice rule describes how to determine a set of preferred options starting from the set of individual preference relations. Thus, it may be informally represented as follows:

$$\{R_1, \ldots, R_N\} \to 2^S$$

For our considerations it is not important if individual preferences are directly given by "real individuals" or, e.g., describe the interests of some social groups. What is important they have to be somehow aggregated so as to produce a set of options satisfying preferences of all involved parties according to some rationality principles. Here, we do not care if there are some intermediate steps in the process of choice. For example, the rule may first require creation of a group (collective) preference relation and only then - using this relation - select a set of options (so-called indirect approach). Moreover, some interesting and popular rules are meant just for producing group preference relations leaving the choice of the "best" options as irrelevant or obvious (e.g., social welfare functions). In cases where produced group preference relations are required to be linear orderings we will assume that options first in ordering are selected.

One of the most popular rules of aggregation is the simple majority rule (known also as the Condorcet rule). Basically, it is assumed to work for linear orderings and produce group linear ordering (what is not always possible). Thus, this rule may be described by the following formulae:

$$R(s_i, s_j) \Leftrightarrow \text{Card}\{k : R_k(s_i, s_j)\} \geq \text{Card}\{k : R_k(s_j, s_i)\} \quad (5)$$

$$S_0 = \{s_i \in S : \underset{i \neq j}{\forall} R(s_i, s_j)\} \quad (6)$$

where Card{A} denotes cardinality of the set A and S_0 the set of collectively preferred options. As a counterpart for this rule in fuzzy case Nurmi [5] proposed the following rule:

$$R(s_i, s_j) \Leftrightarrow k : R_k(s_i, s_j) > \alpha\} \geq threshold \quad (7)$$

$$\{s_i \in S : \neg \underset{j}{\exists} R(s_j, s_i)\} \quad (8)$$

Thus, Nurmi restated (5) adapting it to the fuzzy relation R and employing more flexible concept of majority defined by a threshold. Notice that in (8) still the strict quantifying is used (referring here to the concept of non-domination).

Kacprzyk [2,3] interpreted rule (5)-(6) employing fuzzy majority. He introduced the concept of Q-core that may be informally stated in a slightly modified version ($Q1$/$Q2$-core) [13]:

$CC_{Q1,Q2}$: *Set of options, which are for most (Q2) of individuals "better" than most (Q1) of the rest of options from the set S.*

$$CC_{Q1,Q2} \in F(S)$$

$$\mu_{CC_{Q1,Q2}}(s_i) \rightarrow \underset{s_j}{Q1} \; \underset{x_k \in X}{Q2} \; R_k(s_i, s_j) \quad (9)$$

Then, using Zadeh's linguistic quantifiers:

$$h_i^j = \frac{1}{N}\sum_{k=1}^{N} r_{ij}^k \quad h_i = \frac{1}{M-1}\sum_{\substack{j=1 \\ j \neq i}}^{M} \mu_{Q2}(h_j^k) \quad (10)$$
$$\mu_{CC_{Q1,Q2}}(s_i) = \mu_{Q1}(h_i)$$

where h_i^j denotes the degree to which in opinion of all individuals the option s_i is better than the option s_j; h_i denotes the degree to which in opinion of all experts the option s_i is better than most ($Q1$) other options; $\mu_{Q1}(h_i)$ denotes the sought degree to which in opinion of most ($Q2$) individuals the option s_i is better than most ($Q1$) other options.

Formula (9) serves as a prototype for our generic collective choice rule proposed in the next section.

5. COLLECTIVE CHOICE RULES CLASSIFICATION

It turns out that the $Q1$/$Q2$-core rule may me treated as a generic scheme for many well-known aggregation rules. As all these rules employ, more or less explicitly, only classical quantifiers. Thus, in order to cover them by our generic rule we prefer to use in it OWA operators instead of linguistic quantifiers in the Zadeh sense. Thus, using notation from section 3 we, first transform (9) as:

$$\underset{s_j}{Q1} \; \underset{x_k \in X}{Q2} \; R_k(s_i, s_j) \rightarrow O_{most}^j O_{most}^k R_k(s_i, s_j)$$

In what follows j and k will be indexing the set of options and individuals, respectively. Thus, O_{most}^j (O_{most}^k) denotes an OWA operator aggregating some values for all options (individuals) and governed by the weight vector indicated by the lower index - here *most.*

Now, proposed generic collective choice rule (CCR) may be expressed as follows:

$$\mu_{CCR}(s_i) = O_1 O_2 R_k(s_p, s_q)$$

This scheme has a number of "degrees of freedom". Namely, specific collective choice rules may be recovered deciding:

1. what are the upper indexes of the OWA operators, i.e., if we first aggregate over individuals and then over options or opposite,
2. what are weights vectors of both OWA operators,
3. if the pair of options indexes (p, q) corresponds to (i, j) or to (j, i)

Thus, basically we can distinguish four types of collective choice rules:

I. $\mu_{CCR}(s_i) = O_1^k O_2^j R_k(s_i, s_j)$

II. $\mu_{CCR}(s_i) = O_1^j O_2^k R_k(s_i, s_j)$

III. $\mu_{CCR}(s_i) = O_1^k O_2^j R_k(s_j, s_i)$

IV. $\mu_{CCR}(s_i) = O_1^j O_2^k R_k(s_j, s_i)$

In order to identify the classical rules covered by this generic scheme we have to propose a way to determine a non-fuzzy set of preferred options having a fuzzy set represented by the membership function μ_{CCR}. This may be done in the following way:

- for type I and II rules choose s_i such that
  $\mu_{CCR}(s_i) = \max_j \mu_{CCR}(s_j)$
- for type III and IV rules choose s_i such that
  $\mu_{CCR}(s_i) = \min_j \mu_{CCR}(s_j)$

Now we can point out some well-known rules covered by our generic scheme. In what follows we are using some specific OWA operators as defined at the end of the section 3. Most of these rules assume the individual preferences in the form of linear orderings and we will comment them in these terms.

First, some rules, which may be classified as type I as well as type II:

1. $O_\forall O_\forall$ "consensus solution"
2. $O_{avg} O_{avg}$ Borda

On the other hand, the following rule may be classified as type III or IV:

3. $O_\exists O_\exists$ minimax degree set (Nurmi) [5]

Now some type I rules:

4. $O^k_{avg} O^j_\forall$ plurality voting
5. $O^k_{maj} O^j_\forall$ qualified plurality voting
6. $O^k_{avg} O^j_{maj}$ approval voting-like

provided O^j_{maj} models individuals'. behaviour

O^j_{most} → cumulative variant

7. $O^k_\forall O^j_{maj}$ "consensus+approval voting"

Examples of type II rules follow:

8. $O^j_\forall O^k_{maj}$ simple majority (Condorcet)
9. $O^j_\forall O^k_\exists$ Pareto
10. $O^j_{avg} O^k_{maj}$ Copeland

An example of III rule:

11. $O^k_{most} O^j_{avg}$ Kacprzyk's *Q*-minimax set [2-4]

And finally, some type IV rules:

12. $O^j_\exists O^k_{avg}$ minimax set (Kramer [5])
13. $O^j_\forall O^k_{maj}$ Condorcet-looser
14. $O^j_\exists O^k_\forall$ Pareto inferior options

Thus, the generic scheme covers some classical rules, especially well-known in the context of voting notably corresponding. Some of the recovered rules are not collective choice rules sensu stricto. For example, rules 13 and 14 produce the sets of options that may be treated as collectively rejected rather than selected.

6. CONCLUDING REMARKS

In the paper we proposed a generic scheme of collective choice rule covering many classical rules. The generic scheme proposed makes possible to identify new rules like e.g., rule number 7.

The proposed scheme may give a better insight into the aggregation process. Although some classical rules are defined using crisp quantifiers and individual preference relations the proposed counterpart emerging from our generic scheme may be directly applied to fuzzy preferences.

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