On Coherence Measures for Finite Fuzzy Sets

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Abstract

Comparison tools for fuzzy sets constitute an indispensable topic. Most of these tools deal with fuzzy sets from the view of similarity, ordering and so forth. Here a comparison tool based on an intuitive concept of coherence is presented, and its properties studied and analyzed in depth. It is also shown as these coherence measures are connected to the Fishburn-Yager's ambiguity measures, and two methods for constructing them, from the ambiguity measures point of view firstly, and from the metrics on $[0,1]^m$ secondly, are provided.

1. Introduction. Coherence Measures

Most of the existent comparison tools between fuzzy sets deal with them from a similarity point of view or from an ordering point of view [1]. According to these interpretations, a fuzzy set is extremely similar to itself [1,2]. But similarity measures between fuzzy sets are not sensitive to their respective fuzziness. In fact, the similarity between fuzzy sets near to a crisp set are managed in the same way than the similarity between fuzzy sets more ambiguous. Let us suppose $X=\{x_1,...,x_m\}$ is a finite set, a fuzzy set A on X can be considered as a point of $[0,1]^m$, the more central point of $[0,1]^m$ a fuzzy set is, the more fuzzy it is. Inversely, the more extreme point of $[0,1]^m$ a fuzzy set is, the less fuzzy it is. It is for this reason that a comparison tool for fuzzy sets provides the same result when two central and two extreme points (at the same extreme) are compared, as they are similar in both cases. From this fuzziness point of view, this result is unsatisfactory as central points have less definition than the extreme ones, and its similarity is less meaningful.

Ambiguity and fuzziness measures [3, 4, 5, 6] are tools that allow us to distinguish between these two types of fuzzy sets, but usualy they are applied to isolated fuzzy sets. To define a tool giving information on both concepts, similarity and ambiguity between fuzzy sets at once, here will be defined a type of similarity measures, mapping in [0,1] the cartesian product of the set of fuzzy sets on X by itself, such that at diagonal positions is a measure of crispness. This type of maps will be called coherence measures.

This paper is devoted to present some results concerning these measures, avoiding for the sake of space the details of the proofs, which will be provided in an immediate enhanced version.

Definition 1. Let $X = \{x_1,...,x_m\}$ be a finite set and $P^f(X)$ the set of fuzzy sets on X, we say that

$$cohe: P^f(X) \times P^f(X) \to [0,1]$$

is a *coherence measure* on $P^f(X)$ if and only if:

C1. cohe(A,B) = cohe(B,A)

C2. $cohe(A,B^c) = 1 - cohe(A,B)$

C3. $cohe(\emptyset,X) = 0$

Lemma 2. Let $cohe: P^f(X) \times P^f(X) \to [0,1]$ be a coherence measure then:

a) $cohe(A^c,B^c) = cohe(A,B)$

b) $cohe(\emptyset,\emptyset) = cohe(X,X) = 1$

c) If $A^*(x) = 0.5\ \forall x$, then $\forall A \in P^f(X)$ $cohe(A,A^*) = 0.5$

The following lemma shows a negative result about the monotony of the coherence measures.

Lemma 3. Let

$$cohe: P^f(X) \times P^f(X) \to [[0,1]$$

be a coherence measure. Then neither is true

(1) $\forall A, B, C, D \in P^f(X)$: $A \subseteq B$, $C \subseteq D$

$$cohe(A,C) \le cohe(B,D)$$

nor,

(2) $\forall A, B, C, D \in P^f(X)$: $A \subseteq B$, $C \subseteq D$

$$cohe(A,C) \ge cohe(B,D)$$

2. Coherence and ambiguity

Here the relationship between ambiguity measures [3, 4, 5, 6] and coherence measures will be shown, studied and analyzed in depth. But first one needs to consider the very well known definition of measure of ambiguity for classic sets:

Definition 4. (Fishburn, 1993). Let X be a set, and let P(X) be the set of subsets on X. Then $\alpha: P(X) \to [0,1]$ is an *ambiguity measure* iff the following axioms hold:

A1. $\alpha(\emptyset) = 0$

A2. $\alpha(A) = \alpha(A^c)$

A3. $\alpha(A \cup B) + \alpha(A \cap B) \le \alpha(A) + \alpha(B)$

The extension of this definition to fuzzy sets is straightforwarded (Yager, 1995). Hence, using the usual union, intersection and complement definitions, one obtains

Definition 5. (Yager, 1995). Let X be a set, and $P^f(X)$ the set of the fuzzy subsets on X. Then

$$\alpha: P^f(X)\rightarrow[0,1]$$

is an *ambiguity measure* iff the following axioms hold:

A1. $\alpha(\emptyset) = 0$

A2. $\alpha(A) = \alpha(A^c)$

A3. $\alpha(A\cup B) + \alpha(A\cap B) \leq \alpha(A) + \alpha(B)$

where $(A\cup B)(x) = \max(A(x),B(x))$, $A^C(x) = 1 - A(x)$, and $(A\cap B)(x) = \min(A(x),B(x))$.

and α is called an ambiguity measure in Fishburn-Yager sense.

Lemma 6. Let cohe:$P^f(X)\times P^f(X)\rightarrow[0,1]$ be a coherence measure, then $\alpha:P^f(X)\rightarrow[0,1]$ defined by:

$$\alpha(A) = 1- \text{cohe}(A,A)$$

is an ambiguity measure in the Fishburn-Yager sense if and only if the following expression hold,

$$\text{cohe}(A,A)+\text{cohe}(B,B)\leq\text{cohe}(A\cup B,A\cup B)+\text{cohe}(A\cap B,A\cap B)$$

Then it makes sense to formulate the following question: Let α be an ambiguity measure in the Fishbrun-Yager sense, is it possible to extend α to another coherence measure ß:$P^f(X)xP^f(X)\rightarrow[0,1]$ such that ß(A,A) =1 - α(A)?

We first analyze some counterexamples :

Counterexample 1: Consider $\alpha(A) = 0$, $\forall A\in P^f(X)$. It is obvious that α is an ambiguity[1] measure. If we suppose that ß(A,A) = 1 - α(A) is a coherence, by C2 : ß(A,A^c) = 1 - ß(A,A) = α(A), hence ß(A,A^c) = 0, $\forall A\in P^f(X)$. However, by Lemma 2 c) ß(A^*,A^*) = 0.5, which is contradictory with $A^*=A^{*c}$.

Counterexample 2: Consider α defined by $\alpha(\emptyset) = \alpha(X) = 0$, $\alpha(A) = 1$, $\forall A\in P^f(X)$, $A\neq\emptyset$,X. Then by reasoning similarly as above, it is easy to obtain a contradiction.

As these counterexamples shown, always is not possible to provide the seeked extension. However the following results are looking for finding necessary conditions for the existence of such an extension..

Lemma 7. Let $\alpha:P^f(X)\rightarrow[0,1]$ be an ambiguity. In order to extend α to a coherence measure ß,

$$ß:P^f(X)xP^f(X)\rightarrow[0,1]$$

such that ß(A,A) = 1 - α(A), it is necessary that $\alpha(A^*) = 0.5$, where $A^*(x) = 0.5$, $\forall$x.

[1] For the sake of abbreviation, we use "ambiguity" instead of "ambiguity measure in the Fishburn-Yager sense", and "coherence" instead of "coherence measure".

Lemma 8. Let X be a finite set of m elements, and two sets A= $(a_1 ,...,a_m)$, B = $(b_1 ,...,b_m) \in P^f(X)$, that is $A(x_i) = a_i$ and $B(x_i) = b_i$. Consider ß:$P^f(X)xP^f(X)\rightarrow[0,1]$, defined by:

$$\beta(A,B) = \sum_{i=1}^{m} f(a_i, b_i)$$

Then ß is a coherence measure iff the map f:$[0,1]^2 \rightarrow[0,1]$ verifies:

a) f(x,y) = f(y,x)

b) f(x, 1-y) = (1/m) - f(x,y)

c) f(1-x,y) = (1/m) - f(x,y)

d) f(0,1) = 0

The following results shown some preliminary conclusions

Results 9

1. From b) and a) we deduce that f(x,0.5) = f(0.5,x) = 1/(2m) $\forall x\in[0,1]$

2. f(x,y) $\leq$ (1/m) $\forall x,y\in[0,1]$. Let us suppose the opposite, that is to say, f(x,y) > (1/m), then by b) above:

$$f(x, 1-y) = (1/m) - f(x,y) < (1/m) - (1/m) = 0$$

which is contradictory with f:$[0,1]^2 \rightarrow[0,1]$.

3. From b) above it follows that

$$f(0,0) = f(1,1) = 1/m$$

Hence if A is a crisp set (i.e. only with membership values equal to 0 and 1) directly β(A,A) = 1.

4. If f would have maxima only on (0,0) and (1,1), then by 3, f(x,y) = 1/m if and only if (x,y) = (0,0) or (x,y) = (1,1)

5. In this last case, reciprocaly to 3., it is also easy of showing that if β(A,A) = 1, then A is crisp.

Hence, the following theorem holds:

Theorem 10. Under the same conditions than in lemma 8 above, the following properties hold:

a) If f is monotonic on $[0,0.5]^2$, then it is decreasing on both arguments

b) Moreover,

f is increasing on both arguments on $[0.5,1]^2$.

f is decreasing on x and increasing on y on [0.5,1]x[0,0.5]

f is increasing on x and decreasing on y on [0,0.5]x[0.5,1]

Lemma 11. Let X be a referential finite set with m elements, and A = $(a_1 ,...,a_m)$ and B = $(b_1 ,...,b_m)$ two fuzzy sets on X, where $A(x_i) = a_i$ and $B(x_i) = b_i$. Consider the set A* = (0.5,...,0.5), and let ß:$P^f(X)xP^f(X)\rightarrow[0,1]$ be the coherence measure defined by:

$$\beta(A,B) = \sum_{i=1}^{m} f(a_i, b_i)$$

If f is monotonic on $[0,0.5]^2$ then:

$$\left.\begin{array}{l} A \subseteq B \subseteq A^* \\ A' \subseteq B' \subseteq A^* \end{array}\right\} \Rightarrow \beta(A, A') \geq \beta(B, B')$$

and

$$\left.\begin{array}{l} A^* \subseteq B \subseteq A \\ A^* \subseteq B' \subseteq A' \end{array}\right\} \Rightarrow \beta(A, A') \geq \beta(B, B')$$

This last result gives us the conditions for a desirable property of the coherence measures, which besides confirms an intuitive characteristic: the nearest are two elements A, B from ∅ or X, the greater coherence measure shall be.

The following lemma presents a method for constructing coherence measures starting from metrics. The basis is as follows. As the elements of $P^f(X)$ are maps from X on [0,1], the isomorphism $P^f(X) \approx [0,1]^m$ can be straightforward established, what allows to start from metrics on $[0,1]^m$ in order to construct coherence measures on $P^f(X)$.

Lemma 12. Let X be a finite set of m elements, $P^f(X)$ the set of fuzzy subsets on X, and d: $P^f(X)xP^f(X)\rightarrow[0,1]$ a bounded metric defined by:

$$d(A,B) = (\sum_{i=1}^{m} h(a_i, b_i))^{1/r} \;, r \geq 1$$

Then starting from d, a coherence measure can be constructed as:

$$\beta(A,B) = \frac{1 + d(A, B^c) - d(A,B)}{2}$$

if and only if a) h(0,1) = (1/m), and b) h(a, 1-b) = h(1-a,b) ∀a,b∈[0,1].

This lemma shows also the existence of coherence measures. As illustration, if the lemma is applied to some concrete metrics, one can obtain coherence measures to be compared.

1. As all of the r-metrics (r≥1) on $P^f(X) \cong [0,1]^m$:

$$d(A,B) = (\frac{1}{m} \cdot \sum_{i=1}^{m} |a_i - b_i|^r)^{1/r}$$

verify a) and b), it is obious that a number of coherence measures, depending on r, can be defined from them.

2. Consider h:[0,1]→[0,1] defined as:

$$\begin{cases} h(a,a) = 0 & \forall a \in [0,1] \\ h(a,b) = \dfrac{1}{m} & \forall a,b \in [0,1],\ a \neq b \end{cases}$$

Let I ={1,...,m}, A,B∈$P^f(X)$, G = {i∈I/ $a_i \neq b_i$ }, and let g the cardinal of G. Then

$$d(A,B) = (g/m)^{1/r}$$

obviously holds all of the conditions of the lemma 12, and consequently one can obtain a rough coherence such that

a) for any A such that $a_i \neq 0.5$, ∀i∈I, ß(A,A) = 1, and therefore ß(A,A^c) = 0, and

b) if A(x) ≠ B(x) and A(x) ≠ 1 - B(x), ∀x∈X, then ß(A,B) = 0.5

Next lemma provides us with an ambiguity measures production tool for those ones being defined by sums.

Lemma 13. Let X = {x_1,...,x_m} be a finite set, and α:$P^f(X)\rightarrow[0,1]$, defined by

$$\alpha(A) = \sum_{i=1}^{m} g(a_i)$$

then α is an ambiguity measure (in the Fishburn-Yager sense) if and only if g:[0,1]→[0,1] holds:

a) g(0)=0

b) g(a) = g(1-a)

The next lemma is a consequence of the previous one, and shows that there exist ambiguity measures additive with respect of their component.

Lemma 14. Let P(x) be a polynomial. P(x) holds conditions of lemma 13 if and only if:

$$P(x) = K \cdot (x \cdot (1-x))^{\gamma} \cdot (-x^2 + x + q_1)^{\beta_1} ... (-x^2 + x + q_s)^{\beta_s}$$

where

$$K = \frac{2^{2\cdot\gamma-1}}{m \cdot (q_1 + \frac{1}{4})^{\beta_1} \cdot ... \cdot (q_s + \frac{1}{4})^{\beta_s}}$$

and γ, $ß_i$ are natural numbers (γ≠0).

The following result gives a negative answer to the attempt of extending an ambiguity measure, and in concrte shows that extensions to coherence measures may be not trivial.

Lemma 15. Let X={x_1,...,x_m} be a finite set. Let α:$P^f(X)\rightarrow[0,1]$, an ambiguity measure (in the Fishburn-Yager sense) defined by:

$$\alpha(A) = \sum_{i=1}^{m} g(a_i)$$

Then, there are not any function k:$[0,1]^2\rightarrow[0,1]$ such that:

$$\beta(A,B) = \sum_{i=1}^{m} k(g(a_i), g(b_i))$$

be a coherence measure extending α, i.e., such that

α(A) = 1 - ß(A,A)

Extension Theorem 16. Let X={x_1,...,x_m} be a finite set, and α:$P^f(X)$ → [0,1] an ambiguity measure (in the Fishburn-Yager sense) defined by:

$$\alpha(A) = \sum_{i=1}^{m} g(a_i)$$

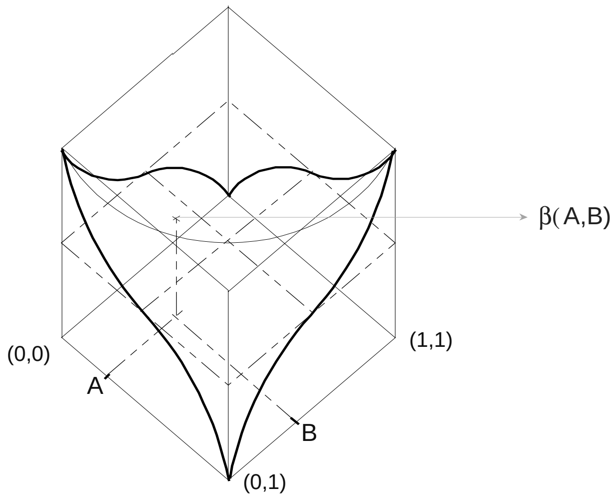
with α(A*) = 0.5. Then, one can define coherence measures

ß:$P^f(X)$ x $P^f(X)$ → [0,1]

such that

ß(A,A) = 1 - α(A)

The figure below gives an outlined graphic representation of the results of the above theorem. To construct this figure the Lemma 15 and the Extension Theorem 16 were used. For the sake of clearity, one supposes that the number of elements of X is m = 2. The basis of the cube ($[0,1]^2$) stands for the set $P^f(X)$. The axis are oriented in such a way that A and B become visibles. Moreover, the height of the cube stands for that value of the coherence measure β(A,B), whereas the diagonal offers β(A,A) = 1 - α(A).



In the following, and as an application of the Extension Theorem, we show how some coherence measures can be obtained from several ambiguity measures.

Application 1. Let consider the ambiguity measure given by Yager [7, 8], although normalized so that can be extended to a coherence measure. Yager's measure is given by the expression:

$$Fuzz(A) = k \cdot \sum_{i=1}^{m} D(a_i)$$

where

$$D(a_i) = \min(a_i, 1 - a_i)$$

By lemma 7, if A^* is such that $A^*(x) = 0.5$ for any x, in order to Fuzz can be extended, it is necessary that $Fuzz_2(A^*) = 0.5$, hence we deduce k = 1/m.

Thus by applying˝the theorem above, one can obtain˝the following coherence measure, denoted $ß_{Ymax}$:

$$\beta_{Ymax}(A,B) = \sum_{i\in J_{11}} (\frac{1}{m} - (\frac{1}{m} \cdot \max(a_i, b_i))) + \sum_{i\in J_{12}} (\frac{1}{m}\max(a_i, 1-b_i)) +$$
$$+ \sum_{i\in J_{21}} (\frac{1}{m}\max(1-a_i, b_i)) + \sum_{i\in J_{22}} (\frac{1}{m} - (\frac{1}{m}\max(1-a_i, 1-b_i)))$$

Similarly, one has the following expression for the coherence˝measure (denoted $ß_{Ymin}$) obtained by using˝the min operator instead of the max one:

$$\beta_{Ymax}(A,B) = \sum_{i\in J_{11}} (\frac{1}{m} - (\frac{1}{m} \cdot \min(a_i, b_i))) + \sum_{i\in J_{12}} (\frac{1}{m}\min(a_i, 1-b_i)) +$$
$$+ \sum_{i\in J_{21}} (\frac{1}{m}\min(1-a_i, b_i)) + \sum_{i\in J_{22}} (\frac{1}{m} - (\frac{1}{m}\min(1-a_i, 1-b_i))) + \sum_{i\in J_*} \frac{1}{2 \cdot m}$$

Application 2. In the same way above, normalizing so that˝can be extended to a˝coherence measure, one can apply the Extension Theorem to˝the ambiguity measure of˝De Luca and Termini [9]. De Luca and Termini's measure is given by

$$Fuzz(A) = -(\sum_{i=1}^{m} a_i \cdot \ln(a_i) + \sum_{i=1}^{m} (1-a_i) \cdot \ln(1-a_i))$$

Normalizing:

$$Fuzz(A^*) = -2 \cdot m \cdot 0.5 \cdot \ln(0.5) = m \cdot \ln 2$$

for this reason˝we re-define˝the ambiguity measure by:

$$Fuzz^*(A) = \frac{1}{2 \cdot m \cdot \ln 2} Fuzz(A)$$

Then,˝by application of the Extension˝Theorem˝to this measure,˝as above one obtaines two coherence˝measures, $ß_{Dmax}$ and $ß_{Dmin}$ whose expressions are omitted for space reasons.

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