FREE MV-ALGEBRAS

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1 Abstract

It is well known that MV-algebras are algebraic models of infinite valued Lukasiewicz logic. We assume the reader's familiarity with MV-algebras.

We recall that an algebra $A = (A; 0, 1, \oplus, \circ, *)$ is said to be an MV-algebra iff it satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $0^* = 1$;
6. $1^* = 0$;
7. $x \circ y = (x^* \oplus y^*)^*$;
8. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Henceforth we shall write ab instead of $a \circ b$ and a^n for $\underbrace{a \circ \cdots \circ a}_{n \text{ times}}$ for given $a, b \in A$. Every MV-algebra has an underlying ordered structure defined by:

The unit interval of real numbers $[0,1]$ endowed with the following operations : $x \oplus y = \min(1, x+y), x \circ y = \max(0, x+y-1), x^* = 1-x$, becomes an MV-algebra. Let Q denotes the set of rational numbers, for $(0 \neq) m \in \omega$ or $m = \omega$ we set $S_m = (S_m; \oplus, \circ, *, 0, 1)$ where

$$\begin{aligned} S_m &= \left\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\right\} \quad \text{if} \quad m \in \omega, \\ S_\omega &= [0,1] \cap Q. \end{aligned}$$

We denote the variety of all MV-algebras by $\mathbb{MV}$. If $\emptyset \neq X \subset \mathbb{MV}$, then $V(X)$ denotes the subvariety generated by X. In these notations we have $\mathbb{MV} = V(S_\omega) = V([0,1])$. The free MV-algebra with m free generators is an algebra of continuous $[0,1]$-valued functions over m-cube $[0,1]^m$ with pointwise MV-operations, introduced by McNaughton. More precisely McNaughton has proved that a function

$$f : [0,1]^m \to [0,1]$$

has an MV polynomial representation $q(x_1, \ldots, x_m)$ such that $f = q$ iff f satisfies the following conditions:

(i) f is continuous,

(ii) there exists a finite number of distinct polynomials $\lambda_1, \ldots, \lambda_s$, each having the form

$$\lambda_j = b_j + n_{j_1} x_1 + \cdots + n_{j_m} x_m$$

where all $b's$ and $m's$ are integers such that for every $(x_1, \ldots, x_m) \in [0,1]^m$ there is j, $1 \leq j \leq s$ such that

$$f(x_1, \ldots, x_m) = \lambda_j(x_1, \ldots, x_m).$$

We call that function *a McNaughton function.*

We analize finitely generated free algebras, presenting the following results:

i) a characterization of finitely generated free MV-algebras as subalgebras of an inverse limit of a chain of order type $\omega^\star$ of finite algebras. It follows that finite algebras are isomorphic to free algebras in locally finite varieties;

ii) a description of the free m-generated MV-algebra as subalgebra of the direct product of all m-generated free MV-algebras of locally finite varieties,

iii) a description of the free m-generated MV-algebra as subalgebra of a direct product of finite chains, by exhibiting the free generators;

iv) a representation of the free m-generated MV-algebra as an MV-algebra of functions from $(S_\omega)^m$ to S_ω, by providing its generators,

v) a representation of the ω-generated free MV-algebra as subalgebra of inverse limit of countably many free algebras, in locally finite varieties,

vi) a representation of the free ω-generated MV-algebra, of a fixed locally finite subvariety of $\mathbb{MV}$, as subalgebra of the inverse limit, of an inverse system, of *finitely many generated* free MV-algebras, in the fixed variety.