QBL: towards a logic for left-continuous t-norms

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Summary

Hájek's BL logic is the fuzzy logic capturing the tautologies of continuous t-norms and their residua. In this paper we investigate a weaker logic, QBL, which is intented to cope with the tautologies of left-continuous t-norms and their residua. The corresponding algebraic structures, QBL-algebras, are defined and completeness of QBL with respect to linearly ordered QBL-algebras is proved.

1 Introduction

Basic Fuzzy logic (BL) is the many-valued residuated logic that was recently introduced by Hájek [4] to cope with the logic of continuous t-norms and their residua. It is built up from two primitive connectives & and $\rightarrow$ and the truth constant $\overline{0}$. The conjunction & is interpreted by a continuous t-norm $*$ and the implication $\rightarrow$ by the residuum of $*$, denoted $\Rightarrow$. Moreover, min-conjunction $\wedge$ and max-disjunction $\vee$ connectives are definable in BL. In [2] it has been finally proved that the theorems of BL are exactly the formulas which are 1-tautologies for each continuous t-norm $*$ and its corresponding $\Rightarrow$.

In this paper we turn our attention to the logic of left-continuous t-norms and present the logic QBL (for quasi Basic logic) as a weakening of BL. Actually, two are the main differences between QBL and BL: (i) the divisibility axiom $\varphi \& (\varphi \rightarrow \psi) \equiv \varphi \wedge \psi$, which is a theorem of BL, is not a tautology for all left-continuous t-norms and their residua, so axiom (A) of BL (which leads to the divisibility condition) is not present in QBL; and (ii), as a consequence of (i), the min-conjunction $\wedge$ is not definable in QBL any more and has to be introduced as primitive connective, together with corresponding axioms. Results we obtain are very analogous of the ones obtained in [4] for BL: most of the main theorems of BL are still provable in QBL, we define the corresponding algebraic structures, called QBL-algebras, and prove a decomposition theorem for them in terms of subdirect products of linearly ordered QBL-algebras, which leads to a completeness theorem of QBL with respect to linearly ordered QBL-algebras.

2 QBL Logic

It is well known that a t-norm defines a residuated implication function (also called residuum) if and only if it is left-continuous (see for instance [3]). Thus, for any left-continuous t-norm $*$, we can define a propositional calculus $QPC(*)$ in the same way that Hájek does in [4] for continuous t-norms, i.e., taking $*$ and its residuum $\Rightarrow$ as the truth functions of (strong) conjunction (&) and implication respectively. The main difference is that the equation $x * (x \Rightarrow y) = min(x, y)$ is not valid for left-continuous t-norms and thus we need to introduce $\wedge$ as a basic connective with & and $\rightarrow$.
In fact the language of the propositional calculus $QPC(*)$ is defined as usual from a countable set of propositional variables $p_1, p_2, p_3, \ldots$, three connectives $\&, \rightarrow, \wedge$ and the truth constant $\overline{0}$. Further definable connectives are:

- $\neg\varphi$ is $\varphi \rightarrow \overline{0}$
- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$

Truth evaluations e are mappings assigning to each propositional variable a truth value belonging to real unit interval [0,1], and are uniquely extended to arbitrary formulas by means of the above mentioned truth functions, that is, requiring:

- $e(\varphi \& \psi) = e(\varphi) * e(\psi)$
- $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$

- $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$

Next we define the QBL system, a weaker system than BL, that aims at capturing the set of 1-tautologies which are common for all $QPC(*)$ calculi, $*$ being a left-continuous t-norms.

Definition 2.1 *Axioms of QBL-Logic are:*

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi \& \psi) \to \varphi$

(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$

(A4) $(\varphi \wedge \psi) \to \varphi$

(A5) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

(A6) $(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$

(A7a) $(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi) \to \chi))$

(A7b) $(\varphi \& \psi) \to \chi)) \to (\varphi \to (\psi \to \chi))$

(A8) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \psi) \to \psi)$

(A9) $\overline{0} \to \varphi$

The rule of inference of QBL is modus ponens.

Notice that axioms A4 and A5 are specific for the new connective $\wedge$ and axiom A6 accounts for only one direction in the previously mentioned divisibility condition. The rest of axioms were already present in BL. Actually, BL is equivalent to the system resulting from QBL by adding the full divisibility axiom

(A6') $(\varphi \& (\varphi \to \psi)) \equiv (\varphi \wedge \psi)$

that makes clear that in BL the conjunction $\wedge$ is definable from & and $\to$.

An easy computation shows that axioms of QBL are provable in BL, and that they are 1-tautologies in each $QPC(*)$ calculus. Moreover, it can be shown that most of theorems proved in [4] for BL (formulas (1)-(35) in [4]) are also provable in QBL. Besides, we can easily prove that the equivalence connective $\equiv$ is also compatible with the conjunction $\wedge$.

The pseudo-deduction theorem valid in BL is also valid in QBL, i.e., given a theory T and formulas φ and ψ, then

$$T \cup \{\varphi\} \vdash \psi \text{ iff there is } n \text{ such that } T \vdash \varphi^n \to \psi$$

where φ^n stands for $\varphi \& \ldots \& \varphi$, n times.

3 QBL-algebras and completeness theorem

From the results of [4] and those of the previous section it seems natural to define the algebras corresponding to the QBL-Logic in the following way.

Definition 3.1 *A QBL-algebra is a residuated lattice $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ satisfying the pre-linearity equation*

$$(x \Rightarrow y) \cup (y \Rightarrow x) = 1.$$

In a similar way Hájek proves in [4] for BL-algebras, we can prove that QBL-algebras form a variety, the only non straightforward matter being, like in the case of BL-algebras, to prove that the residuation condition

$$x * y \leq z \text{ iff } x \leq y \Rightarrow z$$

can be equivalently expressed as a family of equations.

An obvious family of linear QBL-algebras are the algebras defined on the real interval [0,1] by the operations min, max, a left-continuous t-norm $*$ and its residuated implication $\Rightarrow$. On the other hand the quotient of the free algebra of formulas of QBL-Logic with respect to the logical equivalence is also a QBL-algebra. In fact, for each theory T, the relation $\varphi \equiv_F \psi$ iff $T \vdash \varphi \equiv \psi$ is an equivalence relation in the set of formulas of QBL. The corresponding classes will be denoted by $[\varphi]_T$ and the quotient set by L_T. We define

- $0_T = [\overline{0}]_T$
- $1_T = [\overline{1}]_T$
- $[\varphi]_T * [\psi]_T = [\varphi \& \psi]_T$
- $[\varphi]_T \Rightarrow [\psi]_T = [\varphi \to \psi]_T$
- $[\varphi]_T \cap [\psi]_T = [\varphi \wedge \psi]_T$
- $[\varphi]_T \cup [\psi]_T = [\varphi \vee \psi]_T$

We denote the corresponding algebra also by L_T.

Lemma 3.1
With the above definitions, $(L_T, \cap, \cup, *, \Rightarrow, 0_T, 1_T)$ is a QBL-algebra.

Proposition 3.2 *QBL-Logic is complete with respect to the variety of QBL-algebras.*

Proof: Soundness is obvious. Completeness is also an obvious consequence of Lemma 3.1. □

Now we move to the study of filters and prime filters of QBL-algebras and its consequences.

Definition 3.2 *Let L be a QBL-algebra. A filter F is a subset of L satisfying the conditions of BL filter plus*

(f3) for any $x, y \in F$, $x \wedge y \in F$.

A filter is said to be a prime filter iff for any $x, y \in F$, either $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$

Notice that condition (f3) is indeed redundant because it is a consequence of the BL-filter conditions. Namely, since $x * y \leq x \wedge y$, if $x, y \in F$ then $x * y \in F$ and thus $x \wedge y \in F$ as well.

Lemma 3.3 *For any filter F of a QBL-algebra L, define the following equivalence relation in L:*

$$x \sim_F y \text{ iff } x \Rightarrow y \in F \text{ and } y \Rightarrow x \in F$$

Then it holds:
(1) $\sim_F$ is a congruence and the quotient $L/\sim_F$ is a QBL-algebra.
(2) $L/\sim_F$ is linearly ordered iff F is a prime filter.

The proof is an easy translation of the proof of [4, Lemma 2.3.14].

Lemma 3.4 *For any element a of a QBL-algebra L such that $a \neq 1$, there exists a prime filter F not containing a.*

Again, the proof is an easy translation of the proof of [4, Lemma 2.3.15] because it only uses the property (1) of QBL-algebras (property (3) of the definition of BL-algebras in [4]). It follows then a corresponding decomposition theorem for QBL-algebras.

Proposition 3.5 *Each QBL-algebra is a subdirect product of linearly ordered QBL-algebras.*

As a consequence, the lattice structure of a QBL algebra must be distributive as it is proved for provably equivalent formulas in QBL-Logic. Finally, these results lead to the following completeness result.

Theorem 3.6 *QBL is complete, i.e. for each formula φ, the following three statements are equivalent:*
(i) $QBL \vdash \varphi$,
(ii) for each QBL algebra L, φ is a L-tautology,
(iii) for each linearly ordered QBL algebra L, φ is a L-tautology.

4 Final Remarks

This paper is a first step towards a characterization of the logic of left-continuous t-norms. The system QBL presented here results from weakening BL by not requiring the divisibility condition. This leads to introduce the min-conjunction $\wedge$ as a new primitive connective in the logic. Completness of QBL is only shown with respect to linear QBL-algebras, and it remains as future research to investigate completeness of QBL with respect to QBL-algebras of the real unit interval [0, 1]. It also deserves future work to compare QBL with similar many-valued systems as which have been proposed as Höhle's monoidal logics [5], Urquhart's system C [6] and related logics [1].

References

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