NORMAL FORMS FOR FUZZY LOGIC RELATIONS AND THE BEST APPROXIMATION PROPERTY

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Summary

In this paper, we present the representation theorem for fuzzy logic relations, introduce the disjunctive and conjunctive normal forms and formulate what is meant by an approximation of continuous functions by appropriate FL-relations via defuzzification. The best approximation property for the certain choice of defuzzification has been established.

Keywords: Fuzzy logic relations, normal form, defuzzification, approximation of continuous functions.

1 INTRODUCTION

What can be expressed by formulas of predicate fuzzy logic? It is widely declared that fuzzy logic is able to express different kinds of dependences e.g. in control theory, operation research etc., especially when basic events are ill defined. On the other hand, to achieve this goal, only one form of logical expressions is used, namely the "IF–THEN" rules. Thus, on the theoretical level it is interesting to investigate what are the backgrounds for such assertions and in the case they are sound, what are the canonical expressions for description of the represented dependencies.

In this paper, we present the representation theorem for fuzzy logic (FL) relations, which is based on the McNaughton theorem. Then, we introduce the disjunctive and conjunctive normal forms, which formalize the linguistically expressed collections of "IF–THEN" rules. Thus, we come to the class of FL-relations represented by normal forms and see that any continuous FL-relation can be approximated by the one from that class. Finally, an approximation of continuous functions is introduced by applying of a defuzzification operation to the appropriate FL-relation. In this case, we are able to prove the best approximation property. Going back along with the above presentation we see the well justified way of approximate description of continuous functions by fuzzy logic normal forms and thus, by collections of "IF–THEN" rules. The proofs of all the results can be found in the book [1].

Throughout this paper, we will deal with some fixed language J of fuzzy predicate logic, which includes the set of connectives $\{\neg, \nabla, \&, \Rightarrow, \vee, \wedge\}$ and does not include functional symbols. By formula we mean a formula of (fuzzy) predicate logic in the language J. A structure $\mathcal{D}$ for J is formed in accordance with the classical case except for the set of relations, which is replaced by the set of FL-relations. As usually, a fuzzy set is identified with its membership function, and the relation "to be a fuzzy subset of" is denoted by '$\subsetapprox$'.

Furthermore, by a piecewise linear function we mean a continuos function, which is defined and takes values from $[0,1]$ and such that each its linear piece is defined by integer coefficients. Such a function is now usually called the *McNaughton function* (see [2]). The n-dimensional vector $(x_1, \ldots, x_n)$ will be denoted by $\bar{x}^{(n)}$.

Finally, the notion of MV-algebra is well established now and needs not to be specially defined. For the reference see the original paper of C.C. Chang [3]. The most popular example of MV-algebra is the interval $[0,1]$ supplied with the system of Łukasiewicz operations.

2 FL-RELATIONS AND FORMULAS

Definition 1 *Let D be a nonempty set of objects and L be a support of a complete MV- algebra. An n-ary fuzzy logic (FL) relation R on D is a fuzzy set $R \subsetapprox D^n$, $n \geq 0$, given by the membership function $R(\bar{x}^{(n)})$ defined on the set D^n and taking values from*

L.

Furthermore, we will consider the case $L = [0,1]$ and use Łukasiewicz MV-algebra $\mathcal{L}_{\mathrm{L}}$ for the interpretation of logical connectives $\{\neg, \nabla, \&, \Rightarrow, \vee, \wedge\}$ by the operations on $[0,1]$ $\{\neg, \oplus, \otimes, \rightarrow, \vee, \wedge\}$ respectively such that:

$$\neg a = 1 - a,$$
$$a \oplus b = \min(1, a+b), \quad a \otimes b = \max(0, a+b-1),$$
$$a \vee b = \max(a,b), \quad a \wedge b = \min(a,b),$$
$$a \rightarrow b = \min(1, 1-a+b).$$

Definition 2 *Let $A(\bar{x}^{(n)})$ be a formula of fuzzy predicate logic in a language J with a set $FV(A)$ of free variables. Let $\mathcal{D}$ be a structure for J with a domain D and $e : FV(A) \rightarrow D$ be an evaluation of free variables. An n-ary FL-relation $R_A(\bar{x}^{(n)})$ on D is said to be represented by the formula $A(\bar{x}^{(n)})$ with respect to the structure $\mathcal{D}$ and all the possible evaluations e if it establishes a correspondence $R_A : D^n \rightarrow [0,1]$ such that each n-tuple $(\bar{d}^{(n)}) \in D^n$, $d_i = e(x_i)$, $1 \leq i \leq n$, is assigned a value*

$$R_A(\bar{d}^{(n)}) = \mathcal{D}(A_{\bar{x}^{(n)}}[\bar{\mathbf{d}}^{(n)}])$$

where $\bar{\mathbf{d}}^{(n)}$ is a vector of constants corresponding to the vector of elements $\bar{d}^{(n)}$.

Below, we will give the representation theorem ([4]) for FL-relations, which is based on McNaughton theorem ([2]). Since we work with the fixed language J of fuzzy predicate logic, in what following we will use the term "structure" instead of "structure for J".

Theorem 1 *Let $A(\bar{x}^{(n)})$ be a quantifier-free formula and $B_1, \ldots, B_m$ be a list of all pairwise different atomic subformulas of A not being logical constants. Then there exists a piecewise linear function $l(p_1, \ldots, p_m)$ defined on $[0,1]$ such that in any structure $\mathcal{D}$ the FL-relation R_A represented by the formula A is given by the expression*

$$R_A(\bar{x}^{(n)}) = l(f_{B_1}(\bar{x}_1^{(n_1)}), \ldots, f_{B_m}(\bar{x}_m^{(n_m)}))$$

where $f_{B_j}(\bar{x}_j^{(n_j)})$, $1 \leq n_j \leq n$, denotes the membership function of an FL-relation, which is represented by the atomic subformula B_j, $1 \leq j \leq m$, w.r.t. to $\mathcal{D}$. □

As we see, the FL-relations represented by formulas is a subclass of the general class of FL-relations. From the other side, we will show that any FL-relation with continuous membership function can be approximated by some FL-relation represented by a certain formula of fuzzy predicate logic.

3 NORMAL FORMS AND APPROXIMATION THEOREMS

In this section we introduce the disjunctive and conjunctive normal forms and present an approximation theorem for uniformly continuous FL-relations.

Definition 3 *Let $P_1, \ldots, P_k$ be unary predicate symbols and $E_{i_1 \ldots i_n}$, $1 \leq i_j \leq k$, $1 \leq j \leq n$, $n \geq 1$, be some atomic formulas. The following formulas of fuzzy predicate logic are called the MV-disjunctive normal form*

$$\mathrm{DNF}(\bar{x}^{(n)}) = \bigvee_{i_1=1}^{k} \cdots \bigvee_{i_n=1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \,\&\, E_{i_1 \ldots i_n})$$

and the MV-conjunctive normal form

$$\mathrm{CNF}(\bar{x}^{(n)}) = \bigwedge_{i_1=1}^{k} \cdots \bigwedge_{i_n=1}^{k} (\neg P_{i_1}(x_1) \nabla \cdots \nabla \neg P_{i_n}(x_n) \nabla E_{i_1 \ldots i_n}).$$

On the basis of the identity $\neg a \oplus b = a \rightarrow b$, the expression for the MV-conjunctive normal form can be rewritten into the following one

$$\mathrm{CNF}(\bar{x}^{(n)}) = \bigwedge_{i_1=1}^{k} \cdots \bigwedge_{i_n=1}^{k} (P_{i_1}(x_1) \,\&\, \cdots \,\&\, P_{i_n}(x_n) \Rightarrow E_{i_1 \ldots i_n}).$$

We will use the shorts DNF and CNF. In what follows we suppose a support D of a structure $\mathcal{D}$ to be a compact uniform topological space. Let ρ be a pseudometrics, which belongs to the gage of the uniformity on D, i.e. to the family of all pseudometrics which are uniformly continuous on $D \times D$.

Definition 4 *An n-ary FL-relation $R \subsetapprox D^n$ is uniformly continuous on D^n if for any $0 < \varepsilon < 1$ there exists a positive number r, such that if $\rho(x_{11}, x_{21}) < r, \ldots, \rho(x_{1n}, x_{2n}) < r$ then $|R(\bar{x}_1^{(n)}) - R(\bar{x}_2^{(n)})| < \varepsilon$.*

Let us stress that this notion coincides with the analogous classical notion for real valued functions defined on a compact uniform space.

Definition 5 *An FL-relation $S(\bar{x}^{(n)})$ on D is said to ε- approximate another FL-relation $R(\bar{x}^{(n)})$ on D if*

$$|S(\bar{x}^{(n)}) - R(\bar{x}^{(n)})| \leq \varepsilon$$

is true for all $(\bar{x}^{(n)}) \in D^n$.

Theorem 2 *Let an FL-relation $R \subsetapprox D^n$ be uniformly continuous on D^n. Then for any $0 < \varepsilon < 1$, there exist* $\mathrm{DNF}(\bar{x}^{(n)})$ *and a structure $\mathcal{D}$ with the support D such that the FL-relation represented by* DNF *w.r.t. $\mathcal{D}$ ε-approximates $R(\bar{x}^{(n)})$.* □

In predicate fuzzy logic, the formulas $\mathrm{DNF}(\bar{x}^{(n)})$ and $\mathrm{CNF}(\bar{x}^{(n)})$ do not necessarily represent the same FL-relations, even in the same structure. Thus, the approximation property for FL-relations, which are represented by $\mathrm{CNF}(\bar{x}^{(n)})$, must be proved as well.

Theorem 3 *Let an FL-relation $R \subsetapprox D^n$ be uniformly continuous on D^n. Then for any $0 < \varepsilon < 1$, there exist* $\mathrm{CNF}(\bar{x}^{(n)})$ *and a structure $\mathcal{D}$ with the support D such that the FL-relation represented by* CNF *w.r.t. $\mathcal{D}$ ε-approximates $R(\bar{x}^{(n)})$.* □

4 APPROXIMATION OF CONTINUOUS FUNCTIONS VIA DEFUZZIFICATION

As seen, normal forms represent a special type of FL-relations (which can, in particular, approximately represent continuous functions). This representation has an advantage consisting in a finite character of the description by a certain fuzzy logic formula. It would have a real use if the approximating function could be extracted from the formula representing it. This is done by applying the defuzzification procedure.

The specificity of the defuzzification is that it corresponds an *object* with a *fuzzy set*. Therefore, we can regard the defuzzification as a *generalized fuzzy operation* on a set of elements.

Definition 6 *A defuzzification of a fuzzy set $A \subsetapprox X$ given by its membership function $A(x) : X \to [0,1]$ is a mapping $\Theta : \mathcal{F}(X) \to X$ such that*

$$A(\Theta(A)) > 0.$$

The set of all the defuzzification operations on $\mathcal{F}(X)$ will be denoted by $\mathrm{Def}_{\mathcal{F}(X)}$.

Let X be a compact set of reals and $A(x) \not\equiv 0$ be a continuous membership function of a fuzzy set $A \subsetapprox X$. Put $x^* = \Theta(A)$ where $\Theta \in \mathrm{Def}_{\mathcal{F}(X)}$. The following specifications of the defuzzification Θ are used most frequently:

$$x^* = \inf\{x \mid A(x) = \sup_{u\in X} A(u)\}, \tag{1}$$

$$x^* = \frac{x_L + x_G}{2}, \tag{2}$$

$$x_L = \inf\{x \mid A(x) = \sup_{u\in X} A(u)\},$$

$$x_G = \sup\{x \mid A(x) = \sup_{u\in X} A(u)\},$$

$$x^* = \frac{\int_X xA(x)dx}{\int_X A(x)dx}. \tag{3}$$

Now, given a certain defuzzification we can uniquely correspond a function to an FL-relation. Denote the latter by $R(\bar{x}^{(n)}, y)$ and suppose that it is defined on a set $X^n \times Y$. Choose a defuzzification $\Theta \in \mathrm{Def}_{\mathcal{F}(Y)}$ and define the function $f_{R,\Theta}(\bar{x}^{(n)}) : X^n \to Y$ by putting

$$f_{R,\Theta}(\bar{x}^{(n)}) = \Theta(R(\bar{x}^{(n)}, y)). \tag{4}$$

The function $f_{R,\Theta}$ defined by (4) is said to be adjoined to an FL-relation R by means of the chosen defuzzification Θ.

For the simplicity, we will confine ourselves to the case of real valued real functions with one variable.

Definition 7 *Let $f(x)$ be a continuous real valued function defined on the closed interval $[a,b]$ and let $\mathcal{G}$ be a set of approximation, which consists of continuous real valued functions defined also on $[a,b]$. Moreover, let d determine a distance on the set of all continuous real valued functions defined on $[a,b]$. We say that $g^*(x) \in \mathcal{G}$ is the best approximation from $\mathcal{G}$ to f (with respect to d) if the condition*

$$d(f, g^*) \le d(f, g) \tag{5}$$

holds for all $g(x) \in \mathcal{G}$.

We will be interested in finding the best approximation to the given function $f(x)$ in a sense close to the one defined before. As it has been shown earlier, we may regard FL-relations represented by normal forms reduced to

$$\mathrm{DNF}(x,y) = \bigvee_{i=1}^{k} (P_i(x) \,\&\, Q_i(y)) \tag{6}$$

and

$$\mathrm{CNF}(x,y) = \bigwedge_{i=1}^{k} (P_i(x) \Rightarrow Q_i(y)) \tag{7}$$

where P_i and Q_i, $1 \le i \le k$, are unary predicate symbols.

Let $f(x)$ be a continuous function such that $f : [a,b] \to [c,d]$. In order to obtain FL-relations $R_{\mathrm{DNF}}(x,y)$ and $R_{\mathrm{CNF}}(x,y)$, which will be used in the construction of the set of approximation for f we will describe a structure $\mathcal{D}$ dependent on f.

Choose an arbitrary $\varepsilon > 0$ and find $\delta > 0$ so that the inequality $|f(x') - f(x'')| < \varepsilon$ holds true whenever $|x' - x''| < \delta$ and $x', x'' \in [a,b]$. Consider a finite covering of $[a,b]$ by the following half-open δ-intervals, with the exception of the last closed one

$$[a,b] = \bigcup_{i=1}^{k-1} [x_i, x_{i+1}) \cup [x_k, x_{k+1}]$$

where for the simplicity suppose that $k = (b-a)/\delta$ and thus, $x_1 = a$, $x_{k+1} = b$, $x_{i+1} = a + i\delta$, $1 \le i \le k-1$. Denote $I_i = [x_i, x_{i+1})$, $1 \le i \le k-1$, and $I_k = [x_k, x_{k+1}]$. Furthermore, denote by $\bar{f}[I_i] = [f_i, f_{i+1}]$ the closed interval corresponding to I_i, $1 \le i \le k$. Observe that the length of $\bar{f}[I_i]$ is not greater than ε.

Now, put $D = [a,b] \cup [c,d]$ as a support of the structure $\mathcal{D}$ and for each $1 \le i \le k$ assign unary predicate symbols P_i and Q_i fuzzy sets $P_{i,D} \underset{\sim}{\subset} D$ and $Q_{i,D} \underset{\sim}{\subset} D$ given by

$$P_{i,D}(x) = \begin{cases} 1, & \text{if } x \in I_i, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q_{i,D}(y) = \begin{cases} 1, & \text{if } y \in \bar{f}[I_i], \\ 0, & \text{otherwise.} \end{cases}$$

This completes the construction of the structure $\mathcal{D}$ sufficient for the interpretation of the normal forms (6) and (7). It is easy to see that the FL-relations $R_{\mathrm{DNF}}(x,y)$ and $R_{\mathrm{CNF}}(x,y)$ represented by these normal forms w.r.t. $\mathcal{D}$ are equal. We will therefore omit the subscript and denote each of them by $R(x,y)$. The precise expression is

$$R(x,y) = \begin{cases} 1, & \text{if } (\exists i)(1 \le i \le k \,\&\, x \in I_i, y \in \bar{f}[I_i]), \\ 0, & \text{otherwise.} \end{cases} \qquad (8)$$

Finally, define the set of approximation $\mathcal{G}_f$ for f as the set of all functions adjoined to $R(x,y)$ given by (8) by means of the defuzzifications $\Theta \in \mathrm{Def}_{\mathcal{F}(D)}$

$$\mathcal{G}_f = \{g(x) \mid g(x) = f_{R,\Theta}(x) = \Theta(R(x,y)), \quad \Theta \in \mathrm{Def}_{\mathcal{F}(D)}\}. \qquad (9)$$

Lemma 1 *Let $f(x) : [a,b] \to [c,d]$ be a continuous function and $\mathcal{G}_f$ be the set of approximation for f. Then there exists a finite partition $[a,b] = \cup_{i=1}^{k} I_i$, $k \ge 1$, such that $I_1, \ldots, I_k$ are intervals and each function $g(x)$ from $\mathcal{G}_f$ can be characterized as follows:*

$$g(x) = g_i \quad \text{iff} \quad x \in I_i \qquad (10)$$

where $g_i \in \bar{f}[I_i]$ (closure of $f[I_i]$), $1 \le i \le k$.

Moreover, for each k-tuple $(g_1, \ldots, g_k) \in \bar{f}[I_1] \times \cdots \times \bar{f}[I_k]$ there exists a function $g(x) \in \mathcal{G}_f$, which can be characterized by (10).

In what follows, we will show that choosing the defuzzification specified by (2) we obtain the function $g(x) \in \mathcal{G}_f$, which is in some sense the best approximation to the given continuous function $f(x)$. The words "in some sense" are used here because the functions from $\mathcal{G}_f$ are not, in general, continuous.

Let us introduce the set $\mathcal{G}_{f,i}$ of continuous restrictions of functions from $\mathcal{G}_f$ on CI_i as the set of approximation for $f|CI_i$, $1 \le i \le k$. By Lemma 1, for each element $g_i \in \bar{f}[I_i]$, $1 \le i \le k$, there exist a function $g(x) \in \mathcal{G}_f$ such that $g|I_i = g_i$. Thus, $\mathcal{G}_{f,i}$ is simply the set of constant functions defined on CI_i so that $\mathcal{G}_{f,i} = \{g(x) : CI_i \to \bar{f}[I_i] \mid g(x) \equiv g_i,\ g_i \in \bar{f}[I_i]\}$.

Theorem 4 *Let $f(x) : [a,b] \to [c,d]$ be a continuous function and $\mathcal{G}_f$ be the set of approximation for f. Furthermore, let $g(x) \in \mathcal{G}_f$ be the function specified by the defuzzification (2). Then for each closed interval $CI_i = [x_i, x_{i+1}]$, $1 \le i \le k$, the continuous restriction $g|CI_i(x)$ is the best approximation from $\mathcal{G}_{f,i}$ to $f|CI_i(x)$ with respect to the distance d given by*

$$d(u(x), v(x)) = \max_{CI_i} |u(x) - v(x)|. \qquad (11)$$

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