FUZZY ALGEBRAS AS MODELS OF FUZZY THEORIES

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Summary

The main result of this paper is generalization of the classical theorem stating that every submodel of a model of a fuzzy theory is its model iff its axioms are universal formulas. Few consequences for the theory of fuzzy algebras are also presented.

Keywords: Fuzzy logic in narrow sense, model theory, fuzzy algebra.

1 INTRODUCTION

In this paper, we contribute to the model theory of fuzzy logic in narrow sense. The main topic is to demonstrate that analogues of the classical theorems characterizing the relation beween the form of axioms of a theory and the structure of the corresponding class of its models can be proved also in fuzzy logic. Here, we will prove theorem stating that every submodel of a model of a fuzzy theory is its model iff its axioms are universal formulas. Our insipration comes from the book of C. C. Chang and H. J. Keisler [3].

2 PRELIMINARIES

In this paper, we will use the concepts and notation of predicate fuzzy logic in narrow sense presented in the book [7]. By J, we denote a language containing symbols for conjunction $\wedge$, disjunction $\vee$, Łukasiewicz conjunction $\&$, Łukasiewicz disjunction ∇, implication $\Rightarrow$ and negation $\neg$. By F_J we denote a set of all formulas of J, $T = \langle \mathrm{LAx}, \mathrm{SAx}, R\rangle$ is a fuzzy theory where $\mathrm{LAx}, \mathrm{SAx}$ are logical and special axioms being evaluated formulas a/A where $A \in F_J$ and $a \in L$ is its evaluation (hece, axioms can be also considered as fuzzy sets of formulas). The universal formula is a formula of the form $(\forall x)\cdots A$, i.e. all its quantifiers are universal and stay in its begining.

The following extended order relation

$a >^* b$ iff either $b < \mathbf{1}$ and $a > b$, or $a = \mathbf{1}$ and $b = \mathbf{1}$

will be used below.

Theorem 1 (reduction for the consistency) *Let T be a fuzzy theory and $\Gamma \underset{\sim}{\subseteq} F_{J(T)}$ a fuzzy set of formulas. A fuzzy theory $T' = T \cup \Gamma$ is contradictory iff there are $m_1, \dots, m_n$ and $A_1, \dots, A_n \in \mathrm{Supp}(\Gamma)$ such that*

$$T \vdash_c \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$$

and $a_1^{m_1} \otimes \cdots \otimes a_n^{m_n} > \mathbf{0}$ where $c >^* \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$, $a_1 = \Gamma(A_1), \dots, a_n = \Gamma(A_n)$.

By a model, we mean a "structure". If $\mathcal{D}$ is a model of some fuzzy theory then we always explicitly stress this.

Fuzzy theories T_1, T_2 are *equivalent* if one is an extension of the other one. This implies that T_1, T_2 are equivalent iff for every formula $A \in F_J$ we have $T_1 \vdash_a A$ iff $T_2 \vdash_a A$. By the completeness, fuzzy theories are equivalent iff they have the same models.

A model $\mathcal{D}$ is a (fuzzy) submodel of $\mathcal{D}'$, in symbols $\mathcal{D} \subset \mathcal{D}'$, if $D \subseteq D'$, $f_D = f_{D'}|D^n$ holds for each couple of functions $f_D, f_{D'}$ assigned to the functional symbol $f \in J$ and $P_D = P_{D'}|D^n$ holds for each couple of fuzzy relations $P_D \underset{\sim}{\subseteq} D^n$, $P_{D'} \underset{\sim}{\subseteq} (D')^n$ assigned to the predicate symbol $P \in J$. Furthermore, for each pair of the constants u in $\mathcal{D}$ and u' in $\mathcal{D}'$ assigned to a constant symbol $\mathbf{u} \in J$ is $u = u'$.

Let us consider some model $\mathcal{D}$. We extend the language J into $J(\mathcal{D}) = J \cup \{\mathbf{d} \mid d \in D\}$. Furthermore, let a formula $A(x_1, \dots, x_n) \in F_J$ and elements $d_1, \dots, d_n \in D$ be given. Then $A_{x_1,\dots,x_n}[\mathbf{d}_1, \dots, \mathbf{d}_n]$ is the instance of the formula $A(x_1, \dots, x_n)$, in which all the free occurrences of the variables $x_1, \dots, x_n$ have been replaced by the corresponding constants $\mathbf{d}_1, \dots, \mathbf{d}_n \in J(\mathcal{D})$ being assigned to the elements

$d_1, \dots, d_n \in D$, respectively. The model

$$\mathcal{D}_D = \langle \mathcal{D}, \{\mathbf{d} \mid d \in D\} \rangle.$$

is the expanded model of the language $J(\mathcal{D})$.

Given a model $\mathcal{D}_D$, a diagram $\Delta_\mathcal{D}$ is a set of evaluated formulas

$$\Delta_\mathcal{D} = \{ {}^a/P(t_1, \dots, t_n), {}^{\neg a}/\neg P(t_1, \dots, t_n) \mid \\ \mathcal{D}_D(P(t_1, \dots, t_n)) = a, P \in J, t_1, \dots, t_n \in M_V \},$$

i.e. it is a set of evaluated closed atomic formulas and their negations, where the evaluation is equal to the truth degree of the respective formula in the expanded model $\mathcal{D}_D$.

There is also the compactness property, namely a fuzzy theory T has model iff each its finite subtheory $T' \subseteq T$ has a model.

3 THEORIES PRESERVED UNDER SUBMODELS

Lemma 1 *Let T and T' be fuzzy theories in the same language J. Then the following holds true: every model $\mathcal{D} \models T'$ is a model $\mathcal{D} \models T$ iff T' is an extension of T.*

Lemma 2 *Let $\Delta \subseteq F_J$ be a set of evaluated formulas such that $A, B \in \mathrm{Supp}(\Delta)$ implies $A \boldsymbol{\nabla} B \in \mathrm{Supp}(\Delta)$. Let T be a fuzzy theory. Then the following is equivalent.*

- *(a) The fuzzy theory T is equivalent to a fuzzy theory T_Γ with a fuzzy set of axioms $\Gamma \subseteq \Delta$.*
- *(b) Let $\mathcal{D}, \mathcal{E}$ be models such that $\mathcal{D} \models T$ and for every formula $A \in \mathrm{Supp}(\Delta)$, $\mathcal{D}(A) \leq \mathcal{E}(A)$ implies $\mathcal{E} \models T$.*

PROOF: (a)$\Rightarrow$ (b): Let T be equivalent with T_Γ, $\Gamma \subseteq \Delta$. Let $\mathcal{D}, \mathcal{E}$ be models fulfilling the condition of (b). Since T is equivalent with T_Γ, $\mathcal{D} \models T_\Gamma$. However, if $A \in \mathrm{Supp}(\Delta)$ and $T \vdash_a A$ then, from the assumption, $a \leq \mathcal{D}(A) \leq \mathcal{E}(A)$, which implies that $\mathcal{E} \models T_\Gamma$ and due to the equivalence with T, also $\mathcal{E} \models T$.

(b)$\Rightarrow$ (a): Put

$$\Gamma = \{ {}^a/A \mid T \vdash_a A, A \in \mathrm{Supp}(\Delta) \}.$$

We will show that $\mathcal{D} \models T$ iff $\mathcal{D} \models T_\Gamma$.

The implication left-to-right immediately follows from the definition of Γ. To show the opposite implication, let $\mathcal{E} \models T_\Gamma$. We will construct a fuzzy set $\Sigma \underset{\sim}{\subset} F_J$ such that $T \cup \Sigma$ is consistent. Then, by the completeness theorem, there is a model $\mathcal{D} \models T \cup \Sigma$, i.e. $\mathcal{D} \models T$. We find it in such a way that for every $A \in \mathrm{Supp}(\Delta)$, $\mathcal{D}(A) \leq \mathcal{E}(A)$. By the assumption (b) we obtain $\mathcal{E} \models T \cup \Sigma$ and, consequently, $\mathcal{E} \models T$.

Put

$$\Sigma = \{ {}^{\neg b}/\neg B \mid \mathcal{E}(\neg B) = \neg b, B \in \mathrm{Supp}(\Delta) \}.$$

Let $T \cup \Sigma$ be contradictory. By Theorem 1 there are $m_1, \dots, m_n$ and $\neg B_1, \dots, \neg B_n \in \mathrm{Supp}(\Sigma)$ such that

$$T \vdash_c \neg(\neg B_1)^{m_1} \boldsymbol{\nabla} \cdots \boldsymbol{\nabla} \neg(\neg B_n)^{m_n} \quad (1)$$

and $(\neg b_1)^{m_1} \otimes \cdots \otimes (\neg b_n)^{m_n} > \mathbf{0}$ where $c >^* \neg((\neg b_1)^{m_1} \otimes \cdots \otimes (\neg b_n)^{m_n})$. Using the rules of fuzzy logic, the formula in (1) is equivalent to the formula $C := m_1 B_1 \boldsymbol{\nabla} \cdots \boldsymbol{\nabla} m_n B_n$. Then, by the definition of Γ, the evaluated formula ${}^c/C \in \Gamma$. Consequently, we obtain

$$\mathcal{E}(C) = c' \geq c >^* m_1 b_1 \oplus \cdots \oplus m_n b_n.$$

However, since $\mathcal{E}$ is a model, we at the same time have $\mathcal{E}(\neg C) = \neg(m_1 b_1 \oplus \cdots \oplus m_n b_n)$. This implies that $\mathcal{E}(C \otimes \neg C) = c' \otimes \neg(m_1 b_1 \oplus \cdots \oplus m_n b_n) >^* (m_1 b_1 \oplus \cdots \oplus m_n b_n) \otimes \neg(m_1 b_1 \oplus \cdots \oplus m_n b_n) = \mathbf{0}$ — a contradiction. Therefore, $T \cup \Sigma$ is consistent and has a model $\mathcal{D} \models T \cup \Sigma$.

Finally, let $A \in \mathrm{Supp}(\Delta)$ and $\mathcal{D}(A) = a$. We show that $\mathcal{D}(A) \leq \mathcal{E}(A)$. Let $\mathcal{E}(A) = b < a$. Then $\mathcal{E}(\neg A) = \neg b > \neg a$, i.e. ${}^{\neg b}/\neg A \in \Sigma$ and $\mathcal{D}(\neg A) = \neg a \geq \neg b$ since $\mathcal{D} \models T \cup \Sigma$ — a contradiction. □

We say that a fuzzy theory T is *preserved under submodels* if $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{E} \models T$ implies that $\mathcal{D} \models T$.

Theorem 2 *A fuzzy theory T is preserved under submodels iff its special axioms are universal formulas.*

PROOF: Let the special axioms of T be universal formulas. To demonstrate the implication right-to-left, we will suppose that $A := (\forall x)B$ is a universal axiom. Let $T \vdash_a A$, $\mathcal{D} \subset \mathcal{E}$ be models and $\mathcal{E} \models T$. Then

$$a \leq \mathcal{E}((\forall x)B) = \bigwedge_{e \in E} \mathcal{E}(B_x[\mathbf{e}]) \leq \bigwedge_{\substack{e \in D \\ D \subseteq E}} \mathcal{E}(B_x[\mathbf{e}]) = \\ = \bigwedge_{d \in D} \mathcal{D}(B_x[\mathbf{d}]) = \mathcal{D}((\forall x)B). \quad (2)$$

This follows from the fact that $\mathcal{E}(B_x[\mathbf{e}]) = \mathcal{D}(B_x[\mathbf{e}])$ holds for every $e \in D$ (from the definition of submodel using induction for quantifier free formulas).

Conversely, let T be preserved under submodels. Put

$$U = \{ A \mid A, C \in F_J, \models A \Leftrightarrow C, C \text{ is universal} \}.$$

Let $A, B \in U$ be equivalent to universal formulas. Using theorem on prenex form it can be verified that $A \nabla B$ is also equivalent to a universal formula, i.e. $A \nabla B \in U$. Let $\Delta \subsetneqq U$ be a fuzzy set of formulas such that $a/A \in \Delta$ implies that $a/C \in \Delta$ holds for every formula $C \in U$ such that $\models C \Leftrightarrow A$ and $a/A, b/B \in \Delta$ implies $a \oplus b/A\nabla B \in \Delta$. From it follows that if $B_1, \ldots, B_n \in \mathrm{Supp}(\Delta)$ then $m_1 B_1 \nabla \cdots \nabla m_n B_n \in \mathrm{Supp}(\Delta)$ for every $m_1, \ldots, m_n \geq 1$.

Let $\mathcal{D} \models T$ and $\mathcal{D}(A) \leq \mathcal{E}(A)$ for every $A \in \mathrm{Supp}(\Delta)$ where $\mathcal{D}, \mathcal{E}$ are models. We want to demonstrate that $\mathcal{E} \models T$. Then by Lemma 2, this will follow that T is equivalent to T_Γ for some $\Gamma \subseteq \Delta$.

Note that (analogously as in the proof of the analogous classical theorem) $\mathcal{E}(B) \leq \mathcal{D}(B)$ holds for every existential formula B. Let $\Delta_\mathcal{E}$ be a diagram of the model $\mathcal{E}$ and put $T' = T \cup \Delta_\mathcal{E}$.

We will now consider a finite subset $\mathrm{Supp}(\Delta_{\mathcal{E},n}) \subset \mathrm{Supp}(\Delta_\mathcal{E})$ and

$$b_1/B_{1,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m], \ldots, \\ b_n/B_{n,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \in \Delta_{\mathcal{E},n}.$$

Since every instance

$$(B_{1,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \wedge \cdots \wedge B_{n,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \\ \Rightarrow (\exists x_1) \cdots (\exists x_m)(B_1 \wedge \cdots \wedge B_n)$$

is a tautology, we obtain

$$\mathcal{E}(B_{1,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \wedge \cdots \\ \wedge B_{n,x_1\ldots x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m]) \leq \\ \leq \mathcal{E}((\exists x_1) \cdots (\exists x_m)(B_1 \wedge \cdots \wedge B_n)) \leq \\ \leq \mathcal{D}((\exists x_1) \cdots (\exists x_m)(B_1 \wedge \cdots \wedge B_n))$$

and since this inequality holds for every $\Delta_{\mathcal{E},n}$, we conclude that $\mathcal{D} \models \Delta_{\mathcal{E},n}$ and, by the compactness, $\mathcal{D} \models \Delta_\mathcal{E}$. From it follows that $\mathcal{D} \models T'$.

Let $\mathcal{G}_E \models T'$ be a model (expanded by E). Then $\mathcal{G}_E \models T$ and we must show that $\mathcal{E} \subset \mathcal{G}_E$.

Because $\mathcal{G}_E \models \Delta_\mathcal{E}$ and $a/P_{x_1,\ldots,x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m]$, $\neg a/\neg P_{x_1,\ldots,x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m] \in \Delta_\mathcal{E}$ for every closed atomic formula $P_{x_1,\ldots,x_m}[\mathbf{e}_1, \ldots, \mathbf{e}_m]$, we obtain $\mathcal{G}_E(P) = a$ and $\mathcal{G}_E(\neg P) = \neg a$. Consequently, for each two fuzzy relations P_{G_E} and P_E being interpretations of the predicate symbol P in $\mathcal{G}_\mathcal{E}$ and $\mathcal{E}$, respectively

$$P_E = P_{G_E} | E^m.$$

We conclude that $\mathcal{E} \subset \mathcal{G}_E$ (remember that $\Delta_\mathcal{E}$ is a diagram of $\mathcal{E}$). Since $\mathcal{G}_E \models T'$ then $\mathcal{G}_E \models T$ and since T is preserved under submodels, also $\mathcal{E} \models T$. From Lemma 2 it follows that T is equivalent with T_Γ for some $\Gamma \subseteq \Delta$. □

4 FUZZY ALGEBRAS

Theorem 2 proved in the previous section has some consequences, which generalize properties of various fuzzy algebras (e.g. fuzzy groups, lattices, modules, etc.) discussed in the literature (see [1, 10, 11]).

Let us consider, e.g. the concept of fuzzy group introduced alrady by A. Rosenfeld in [10]. A fuzzy group is a fuzzy subset $G \subsetneqq U$ of a group $\langle U, \circ \rangle$ such that

$$G(x) \wedge G(y) \leq G(x \circ y) \\ G(x) \leq G(x^{-1}), \qquad x, y \in U.$$

It can be then proved, for example, that G is a fuzzy group iff every a-cut G_a is a group and also, that intersection of fuzzy groups is a fuzzy group. We want to demonstrate that these results are general and follow from the properties of fuzzy logic and models in it. The fuzzy model theory thus takes a similar role in fuzzy mathematics as classical model theory has in classical mathematics. Moreover, since fuzzy logic (and its model theory) is a generalization of classical logic, this also demonstrates that generalizations of classical results are often obvious consequence of fuzzy model theory.

We will now define a *fuzzy theory of fuzzy algebras.* Its language J consists of equality $=$, n_m-ary functional symbols $o_1, \ldots, o_m$ and a unary predicate predicate symbol G. A fuzzy theory of fuzzy algebras has a set of special axioms

$$\mathbf{1}/A \tag{3}$$

$$\mathbf{1}/(\forall x_1) \cdots (\forall x_n)((G(x_1) \wedge \cdots \wedge G(x_n)) \Rightarrow \\ G(o_j(x_1, \ldots, x_n)), \qquad j = 1, \ldots, m. \tag{4}$$

Moreover, the equality symbol $=$ is always interpreted as sharp, i.e. every formula $t = s$ is interpreted by $\mathcal{D}(t = s) = \mathbf{1}$ in every model $\mathcal{D}$.

The axiom (4) states that G represents a fuzzy set which is closed with respect to the operation being interpretation of o_j. As consequence of Theorem 2, if all axioms (3) are universal formulas then T is preserved under submodels. We will demonstrate this below.

Let $J = \{\circ, G\}$ (the equality symbol is not explicitly stressed). Then the fuzzy theory T with the only closeness axiom

$$\mathbf{1}/(\forall x)(\forall y)(G(x) \wedge G(y)) \Rightarrow G(x \circ y)) \tag{5}$$

is the theory of fuzzy groupoids. Let

$$\mathcal{D} = \langle D, \circ, G_D \rangle \models T$$

(the interpretation of $\circ$ in $\mathcal{D}$ is denoted by the same symbol). Obviously, $\langle D, \circ \rangle$ is a classical groupoid. At the same time, $G_D \subsetapprox D$ is a fuzzy groupoid since it fulfils the inequality

$$G_D(d_1) \wedge G_D(d_2) \leq G_D(d_1 \circ d_2), \qquad d_1, d_2 \in D$$

being interpretation of the closeness axiom (5) in $\mathcal{D}$. Since every a-cut $G_{D,a} \subseteq D$ obviously also fulfils this inequality, it is groupoid, as well. Let us put

$$\mathcal{D}_a = \langle D_a, \circ, G_D | D_a \rangle. \qquad (6)$$

It is a submodel of $\mathcal{D}$ and by Theorem 2, it is a fuzzy groupoid. However, if we extend T by further axioms (3), e.g. associativity, unit element, inverse operation, etc., which are universal formulas, we obtain a fuzzy theory of fuzzy algebra (fuzzy semigroup, group, etc.) which is preserved under submodels. Consequently, every model (6) is a fuzzy algebra.

Moreover, when omitting the closeness axioms (4), we obtain classical theory of classical algebras. They are classically preserved under submodels and thus, the restricted models $\mathcal{D}'_a = \langle D_a, \circ \rangle$ are classical algebras.

Similarly, let $\mathcal{D}, \mathcal{E}$ be fuzzy algebras (models of a fuzzy theory of fuzzy algebra). Then

$$\mathcal{D} \cap \mathcal{E} = \langle D \cap E, P_D \cap P_E, o_1 | D \cap E, \ldots, o_1 | D \cap E \rangle$$

is a submodel of both $\mathcal{D}$ as well as $\mathcal{E}$ and therefore, a fuzzy algebra.

Another similar result is immedate: Let us extend the language J by another unary predicate H, and let us extend the fuzzy theory T to a fuzzy theory T' by the closeness axioms for H so that its interpretation $H_D \subsetapprox D$ forms a fuzzy algebra as well. It can be easily proved in T' that

$$T' \vdash (\forall x_1) \cdots (\forall x_n)(((G(x_1) \wedge H(x_1)) \wedge \cdots \\ \wedge (G(x_n) \wedge H(x_n))) \Rightarrow \\ (G(o_j(x_1, \ldots, x_n) \wedge H(o_j(x_1, \ldots, x_n))).$$

Consequently, intersection of fuzzy algebras is a fuzzy algebra.

5 CONCLUSION

In this paper, we have continued the fuzzy model theory and proved generalization of the classical theorem stating that every submodel of a model of a fuzzy theory is its model iff its axioms are universal formulas. This theorem has consequences for various developments in fuzzy mathematics where various presented results become immediate consequence of it. We have demonstrated this on the example of fuzzy groups but this holds for all kinds of fuzzifications of classical algebras. Therefore, when developing fuzzy mathematics, more specific properties, which either have no counterpert, or have no sense in classical mathematics should be searched.

It may be expected that also generalization of other classical result stating that a fuzzy theory is preserved under unions of sequences of models iff it has universally-existential axioms holds analogously true. This will be the topic of another paper.

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