SYSTEMS OF ORDINAL FUZZY LOGIC WITH APPLICATION TO PREFERENCE MODELLING

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Summary

In this paper, we survey several many-valued propositional logics in which the truth-functions (in the real unit interval $[0,1]$) of their connectives are definable only from the natural ordering of the scale. The usefulness of this logical framework for the formalization of ordinal preference modelling is also shown.

1 INTRODUCTION

The most elementary notion behind a fuzzy set $A : U \to [0,1]$ is that of specifying an ordering $\leq_A$ on U, $u \leq_A v$ iff $A(u) \leq A(v)$, accounting for the compatibility of the elements of U with respect to the fuzzy predicate A. The only thing which is needed for this is the natural ordering of the scale $([0,1], \leq)$. Moreover, this ordering is also sufficient for introducing Zadeh's original fuzzy set theoretical intersection and union operations by means of pointwise min and max, respectively. Finally, in order to define fuzzy set complementation an involutive negation (decreasing transformation of $[0,1]$) is also needed, the most simple one (standard negation) being defined by $n(x) = 1 - x$. Hence, the De Morgan structure $([0,1], \leq, \min, \max, 1-x)$ is the basic truth-value algebra for ordinal fuzzy logic systems.

In Section 2 we survey several systems of ordinal fuzzy logic, all arising from the well-known infinitely valued Gödel logic G, whose truth-functions are definable only from the natural ordering of the real unit interval $[0,1]$, and thus, without using any richer algebraic structure. Typical ordinal connectives are min conjunction $\wedge$, max disjunction $\vee$ and Gödel implication $\to$. Gödel logic G, its extension with rational truth values, RGL, and the extension of G with an involutive negation $\sim$, $G_\sim$, are briefly recalled in Section 2. Two aspects of $G_\sim$ are further elaborated in the two remaining sections of the paper. Deduction with weighted $G_\sim$-clauses is studied in Section 3, and finally, in Section 4 we show how $G_\sim$ can be used as a logical framework for formalizing preference structures in an ordinal setting.

2 GÖDEL LOGIC AND SOME EXTENSIONS

Following [6], the language of propositional Gödel fuzzy logic (denoted hereafter G) is built in the usual way from a (countable) set of propositional variables, a conjunction $\wedge$, an implication $\to$ and the truth constant $\overline{0}$. Further connectives are defined as follows:

$$\begin{array}{lll} \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi & \text{is} & \varphi \to \overline{0}, \\ \varphi \equiv \psi & \text{is} & (\varphi \to \psi) \wedge (\psi \to \varphi). \end{array}$$

The semantics of G is given by *evaluations* e of the propositional variables into the real unit interval $[0,1]$ which are extended to arbitrary formulas by means of the following rules:

$$\begin{aligned} e(\overline{0}) &= 0 \\ e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\ e(\varphi \to \psi) &= \begin{cases} 1, & \text{if } e(\varphi) \leq e(\psi) \\ e(\psi), & \text{otherwise} \end{cases} . \end{aligned}$$

For the derived connectives the truth-evaluations take these forms:

$$\begin{aligned} e(\varphi \vee \psi) &= \max(e(\varphi), e(\psi)) \\ e(\neg\varphi) &= \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases} \\ e(\varphi \equiv \psi) &= \begin{cases} 1, & \text{if } e(\varphi) = e(\psi) \\ \min(e(\varphi), e(\psi)), & \text{otherwise} \end{cases} . \end{aligned}$$

The following is an axiomatization[1] of G:

[1] Actually, axioms (A1) ... (A7) are axioms of Hájek's Basic Logic (BL) [6], and thus it becomes clear that G is indeed the extension of BL with the axiom (A8), requiring the idempotency of the conjunction.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \wedge \psi) \to \varphi$
(A3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A4) $(\varphi \wedge (\varphi \to \psi)) \to (\psi \wedge (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$
(A5b) $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\overline{0} \to \varphi$
(A8) $\varphi \to \varphi \wedge \varphi$

The *deduction rule* of G is modus ponens. The notion of proof is as usual. *Completeness* for G reads as follows: φ is provable in G, written $\vdash \varphi$, iff $e(\varphi) = 1$ for any G-evaluation e. Furthermore, G enjoys *strong completeness* as well. Namely, let T be an arbitrary theory over G, i.e. just a set of formulas. An evaluation e is a model of T if $e(\psi) = 1$ for all $\psi \in T$. Then, T proves φ, written $T \vdash \varphi$, iff $e(\varphi) = 1$ for any G-evaluation e which is a model of T.

An interesting extension of Gödel logic G is the so-called Rational Gödel logic (RGL for short), where, for each rational $r \in [0,1]$, a truth constant $\overline{r}$ is introduced into the language[2]. G-evaluations e are then extended to RGL formulas by requiring $e(\overline{r}) = r$, for all rational $r \in [0,1]$. Axioms of RGL are those of G plus the following book-keeping axioms for truth-constants:

(RGL1) $\overline{r} \wedge \overline{s} \equiv \overline{\min(r,s)}$,
(RGL2) $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$,

where $\Rightarrow$ is Gödel's implication function. For RGL we cannot expect a Pavelka-style completeness (provability degree = truth degree), since this style of completeness strongly relies on the continuity of the truth functions, and it is obvious that in RGL, the truth function of implication is not continuous. Nevertheless one can show the following *classical* completeness (provable = true in all models) result[3].

Theorem 1 (Completeness of RGL) *Let T be a finite theory over* RGL*. Then T proves φ iff $e(\varphi) = 1$ for each evaluation e which is a model of T. In particular, T proves $\overline{r} \to \varphi$ iff $e(\varphi) \geq r$ for every evaluation e which is a model of T.*

One potential drawback of Gödel logic to be used for approximate reasoning is that it lacks an involutive negation. In [4] the authors study residuated logics arising from extending SBL logic (BL logic with Gödel negation) with a new involutive negation connective $\sim$. Particularizing to G, the new resulting logic $G_\sim$ has the axioms of G plus:

($\sim$1) $(\sim\sim\varphi) \equiv \varphi$
($\sim$2) $\neg\varphi \to \sim\varphi$
($\sim$3) $\Delta(\varphi \to \psi) \to \Delta(\sim\psi \to \sim\varphi)$
(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

where $\Delta\varphi$ is $\neg\sim\varphi$. Deduction rules of $G_\sim$ are *modus ponens* and *necessitation* for Δ: from φ derive $\Delta\varphi$.

Notice that the connective Δ (originally introduced and axiomatized in [1]) is actually a two-valued connective: $e(\Delta\varphi) = 1$ if $e(\varphi) = 1$, $e(\Delta\varphi) = 0$ otherwise. $G_\sim$ has been shown to be complete w.r.t. to the so-called standard $G_\sim$-algebra, the unit interval $[0,1]$ equipped with Gödel's truth functions and with the involutive negation $\sim x = 1 - x$.

Theorem 2 [4] **(Standard completeness of $G_\sim$)**
(i) For each $G_\sim$-formula φ, $G_\sim$ proves φ iff φ is a tautology over the standard $G_\sim$-algebra.
(ii) Let T be a theory over $G_\sim$. Then $T \vdash \varphi$ iff $e(\varphi) = 1$ for any evaluation e over the standard $G_\sim$-algebra which is a model of T.

The extension of $G_\sim$ with rational truth-constants has also been considered in [4] by adding to $G_\sim$ the previously considered book-keeping axioms (RGL1) and (RGL2). Completeness results are then also preserved.

3 ON THE IMPLICATION FREE FRAGMENT OF $G_\sim$: A LOGIC OF min, max AND 1-x

Let us consider the sublanguage of $G_\sim$ built from the set of propositional variables and with the connectives $\wedge$, $\vee$ and $\sim$, i.e. without using the implication. Let us denote this sublanguage by $\mathcal{L}_Z$. Let us introduce truth constants by considering weighted formulas as pairs

$$(\varphi, r)$$

where $\varphi \in \mathcal{L}_Z$ and r is a rational from [0, 1]. The intended semantics for this type of weighted formulas is the obvious one: a $G_\sim$-evaluation e will satisfy a formula (φ, r), written $e \models (\varphi, r)$, iff $e(\varphi) \geq r$.

We are interested in defining a complete proof system for $\mathcal{L}_Z$, in the spirit of the pioneering work of R.C. Lee with fuzzy prologs dealing with weighted (Horn) clauses[4]. To this end, let us define a *weighted $\mathcal{L}_Z$-clause* as a weighted disjunction of literals (either positive or negative) and a *weighted theory* as a set of weighted $\mathcal{L}_Z$-formulas.

[2] In the same spirit as what was originally done by Pavelka with the infinitely-valued Łukasiewicz logic and later improved by Hájek, see [6] for details.

[3] This theorem and its proof are an easy adaptation of an analogous theorem in [4].

[4] A tableaux system for arbitrary (non-clausal) $\mathcal{L}_Z$ formulas is already described in [7, Chapter 5].

Proposition 1 (Clausal transformation) *There is a transformation $*$ which maps each weighted theory T into a theory T^* consisting only of weighted $\mathcal{L}_Z$-clauses which is semantically equivalent to T. Moreover, it can be assumed that the clauses do not contain repeated literals.*

In other words, for each weighted theory T and each weighted $\mathcal{L}_Z$-formula (φ, α), we have

$$T \models (\varphi, \alpha) \quad \text{iff} \quad T^* \models (C_i, \alpha_i) \text{ for any } i = 1, \ldots, n.$$

where $\{(\varphi, \alpha)\}^* = \{(C_1, \alpha_1), \ldots, (C_n, \alpha_n)\}$. Thus, we can restrict ourselves to deal with weighted $\mathcal{L}_Z$-clauses. As a matter of fact, most automated proof systems developed for many-valued logics (see for instance [8] for a survey) use the so-called *signed formulas* representation language. Given an scale of truth-values V, a *signed literal* is of the form $S{:}p$, where $S \subseteq V$ is called the *sign*, and it has the following intended meaning: $S{:}p$ is *true* if the propositional variable p takes a truth value belonging to S, $S{:}p$ is *false* otherwise. A sign is called *regular* if it is an upper or lower semi-interval of V. Upon propositional variables and regular signs, one can build a propositional language of clauses of the form

$$S_1{:}p_1 \vee \ldots \vee S_n{:}p_n,$$

with the usual Boolean semantics for the disjunction. Call this language $\mathcal{L}_S$. The basic inference rule for such a kind of signed logic is the following SIGNED_RESOLUTION rule:

$$\frac{S_1{:}p \vee R{:}q, \quad S_2{:}p \vee W{:}r}{R{:}q \vee W{:}r} : \text{if } S_1 \cap S_2 = \emptyset$$

It is shown in [9] that this rule allows a complete deduction calculus by refutation within $\mathcal{L}_S$, complete with respect to the obvious (quasi Boolean) semantics. If we denote by $\vdash_{SR}$ the deduction by repeated use of the Signed_Resolution rule, we have for any (finite) set Γ of $\mathcal{L}_S$ clauses,

$$\Gamma \models S_1{:}p_1 \vee \ldots \vee S_n{:}p_n \quad \text{iff}$$
$$\Gamma \cup \{\overline{S_1}{:}p_1, \ldots, \overline{S_n}{:}p_n\} \vdash_{SR} \bot$$

where $\overline{S_i}$ stands for the complement of S_i and $\bot$ denotes the empty clause. We will denote by $\vdash^r_{SR}$ this notion of deduction by refutation.

Now let us come back to our weighted clauses in Gödel logic with involution. Since for any $\alpha, \beta, \delta \in [0,1]$, $\max(\alpha, \beta) \geq \delta$ iff either $\alpha \geq \delta$ or $\beta \geq \delta$, clauses like $(p \vee q, \alpha)$ or $(\sim p \vee q, \alpha)$ can be equivalently expressed as the following disjunctions of (regular) signed literals:

$$[\alpha, 1]{:}p \vee [\alpha, 1]{:}q, \quad [0, 1-\alpha]{:}p \vee [\alpha, 1]{:}q,$$

respectively. Therefore, deduction with weighted clauses can be performed by translating them into signed formulas. Namely, if we denote by $^\diamond$ the transformation from $\mathcal{L}_W$ to $\mathcal{L}_S$, we have, for any weighted clause (C, α) and theory Δ over $\mathcal{L}_W$ the following completeness result:

$$\Delta \models (C, \alpha) \quad \text{iff} \quad \Delta^\diamond \vdash^r_{SR} (C, \alpha)^\diamond.$$

Therefore, one can define a notion of proof in the language of weighted $\mathcal{L}_Z$-formulas by moving (through the transformation mappings * and $^\diamond$) to the language of signed clauses:

$$T \vdash_Z (\varphi, \alpha) \quad \text{if}_{Def} \quad (T^*)^\diamond \vdash^r_{SR} (C_i, \alpha_i)^\diamond, \forall i = 1, \ldots, n$$

where $\{(\varphi, \alpha)\}^* = \{(C_1, \alpha_1), \ldots, (C_n, \alpha_n)\}$. As a direct consequence of all previous remarks, we have the following completeness condition for $\vdash_Z$:

Theorem 3 $T \models (\varphi, \alpha)$ *iff* $T \vdash_Z (\varphi, \alpha)$.

4 $G_\sim$ AND FUZZY PREFERENCE MODELLING: MAXITIVE FPS

We briefly discuss the usefulness of the system $G_\sim$ as a logical framework for ordinal preference modelling.

A *fuzzy preference structure* (FPS) [2] is a sixtuple $(\mathcal{A}, P, I, J; T, S)$ where $\mathcal{A}$ is the set of alternatives, P, I, J are binary fuzzy relations[5] on $\mathcal{A}$, T is a t-norm, and S is its dual t-conorm. Among others, the following fundamental conditions are required:

(i) disjointness: $X \cap_T Y = \emptyset$, for any two different X and Y in $\{P, P^t, I, J\}$[6];
(ii) completeness: $\overline{P \cup_S I} = P^t \cup_S J$;
(iii) Assignment Principle (AP): at least one of the four degrees (P, P^t, I, J) can be chosen freely.

Among continuous t-norms and t-conorms, only the Łukasiewicz ones satisfy AP [10], leading to *additive* FPS. As a consequence, min is not suitable for the ordinal framework. However, the nilpotent minimum [5], which is only left-continuous, is suitable [2, 3], leading to *maxitive* FPS [3].

Now, since nilpotent minimum is definable in $G_\sim$, we can formalize maxitive FPS as a theory in $G_\sim$. Define two new connectives $\wedge_{nil}$ and $\vee_{nil}$, corresponding to nilpotent minimum and nilpotent maximum (see [5]) as follows:

$$p \wedge_{nil} q \quad \text{is} \quad p \wedge q \wedge \sim\Delta(p \to \sim q),$$
$$p \vee_{nil} q \quad \text{is} \quad \sim(\sim p \wedge_{nil} \sim q).$$

[5] $P(a,b)$ ($P(b,a)$, $I(a,b)$ and $J(a,b)$ resp.) denotes the degree to which alternative a is better than (worse than, indifferent to and incomparable to, resp.) alternative b.

[6] P^t is the transpose of P.

Further, for any $(a,b) \in \mathcal{A}^2$, $a \neq b$, we consider three many-valued propositional variables:

p_{ab}	:	alternative a is better than alternative b
i_{ab}	:	alternatives a and b are indifferent
j_{ab}	:	alternatives a and b are incomparable.

Then we consider the theory $T_{\mathcal{A}}$ over $G_{\sim}$ consisting of $i_{ab} \equiv i_{ba}$ and $j_{ab} \equiv j_{ba}$ and the following formulas:

$$\neg(p_{ab} \wedge_{nil} p_{ba}), \quad \neg(p_{ab} \wedge_{nil} i_{ab}), \quad \neg(p_{ab} \wedge_{nil} j_{ab}),$$
$$\neg(i_{ab} \wedge_{nil} j_{ab}), \quad \sim(p_{ab} \vee_{nil} i_{ab}) \equiv p_{ba} \vee_{nil} j_{ab}.$$

In this framework AP is satisfied, and thus $T_{\mathcal{A}}$ can be used to describe and reason about maxitive FPS. Following [2], consider the definition of non-strict preference

$$r_{ab} \text{ is } p_{ab} \vee_{nil} i_{ab}. \quad (1)$$

Now, one can show that if we add $\neg(p_{ab} \wedge p_{ba})$ (min-asymmetry of strict preferences) to $T_{\mathcal{A}}$ then the extended theory $T^+_{\mathcal{A}}$ proves, over $G_{\sim}$, the formula

$$(r_{ab} \vee r_{ba}) \vee \neg\Delta(\sim r_{ab} \to r_{ba}) \vee (r_{ba} \equiv r_{ab}) \quad \text{(C1)}$$

showing that in a maxitive FPS the possible evaluations of r_{ab} and r_{ba} are indeed restricted: either $\max(e(r_{ab}), e(r_{ba})) = 1$, or $e(r_{ab}) = e(r_{ba})$ or $e(r_{ab}) < 1 - e(r_{ba})$.

Construction of maxitive FPS. We consider the language over $G_{\sim}$ built from the propositional variables

r_{ab} : alternative a is at least as good as alternative b

for any $(a,b) \in \mathcal{A}^2$, $a \neq b$, and introduce the following definitions:

$$\begin{array}{lll} p_{ab} & \text{is} & r_{ab} \wedge_{nil} \sim r_{ba} \\ i_{ab} & \text{is} & r_{ab} \wedge r_{ba} \\ j_{ab} & \text{is} & \sim r_{ab} \wedge \sim r_{ba}. \end{array}$$

Then one can check that C1 is enough to recover all conditions described by the theory $T^+_{\mathcal{A}}$, i.e., $\text{C1} \vdash \varphi$, for any $\varphi \in T^+_{\mathcal{A}}$. Moreover, we recover the former definitions

$$\text{C1} \vdash r_{ab} \equiv (p_{ab} \vee_{nil} i_{ab}).$$

If we introduce $\varphi \lhd \psi$ as $\Delta(\varphi \to \psi) \wedge \neg\Delta(\psi \to \varphi)$[7], one can verify that

$$\text{C1} \vdash (\neg\neg p_{ab} \wedge (p_{ab} \lhd \sim i_{ab})) \to (i_{ab} \lhd p_{ab}).$$

We shall call this last formula[8] C2.

[7] Semantically, $e(\varphi \lhd \psi) = 1$ if $e(\varphi) < e(\psi)$, $e(\varphi \lhd \psi) = 0$ otherwise.

[8] An evaluation e makes C2 1-true whenever $0 < e(p_{ab}) < 1 - e(i_{ab})$ implies $e(i_{ab}) < e(p_{ab})$.

Characterization of maxitive FPS. Conversely, let us start now with the language built up from the propositional variables p_{ab}, i_{ab} and j_{ab}, and introduce propositions r_{ab} as abbreviations like in (1). Now it is condition C2 which allows us to recover the definitions used in the construction. Namely, if we let $T^{++}_{\mathcal{A}} = T^+_{\mathcal{A}} \cup \{\text{C2}\}$, then one can show that $T^{++}_{\mathcal{A}}$ proves

$$\begin{array}{lll} p_{ab} & \equiv & (r_{ab} \wedge_{nil} \sim r_{ba}), \\ i_{ab} & \equiv & (r_{ab} \wedge r_{ba}), \\ j_{ab} & \equiv & (\sim r_{ab} \wedge \sim r_{ba}), \end{array}$$

which nicely closes the *almost* equivalency between the two views on maxitive fuzzy preference structures.

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