AN OVERVIEW OF STABILITY ANALYSIS OF MIMO FUZZY CONTROL SYSTEMS

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Summary

This paper presents a survey on the existing methods for the stability analysis of fuzzy control systems including both conventional Mamdani's rule-based fuzzy controllers and Takagi-Sugeno controllers. The overview considers several approaches including input-output stability, frequency response methods, Lyapunov techniques and qualitative methods. The paper also sumarizes the new stability techniques developed in the "Fuzzy Algorithms for the control of Multi-Input Multi-Output process" (FAMIMO) ESPRIT Project.

Keywords: Stability, Fuzzy Control, Lyapunov Theory, Input-Output methods, Frequency Response, Robust Stability.

1 INTRODUCTION

Stability is a fundamental property of feedback control systems. Practical control engineering have been always concerned with stability problems. In fact, before efforts are made to satisfy performance control criteria, it is imperative that the stability is assured.

The stability properties of linear feedback systems are well understood. The analysis and design techniques of linear feedback control systems are directly or indirectly related to their stability properties. Stability is usually considered in control engineering as a relative concept associated to the dynamic response of a system. It is said that high potential of instability exists if the response exhibit high overshoot, the decay ratio of the oscillations is low (oscillations not damped), and the rise time is fast.

Nonlinear systems stability is a complex problem and is very difficult to derive general results. The practical stability studies of fuzzy logic control systems have been carried out by experimentation in several working conditions. However, in many applications, particularly when safety is an important issue, that is not enough and more general results are required. That motivated many research studies from the seventies. Most results are related to local stability around an equilibrium point. Only some authors have studied the global stability involving all the space in which the variables associated to the process to be controlled can vary.

Most fuzzy control systems have been designed from the expert control knowledge expressed as a set of fuzzy rules. Some heuristic relations between the stability and the fuzzy rules of a closed loop fuzzy controller have been pointed out by several authors. These relations can be analysed by comparing the space of the controller input variables with the trajectories of the dynamic evolution in this space. Adjusting the fuzzy rules after observing the system response curve has been an usual way to improve the stability. That involves the analysis of the effect of the rules in the closed loop step response.

On the other hand, it should be noted that the design of fuzzy control systems from single rules is difficult when the human operator only can verbalize few trivial rules or when the process is too complex. In many applications it is more practical to try to learn automatically the fuzzy rules from experiments. This approach has demonstrated to be useful in many industrial applications. An usual technique to implement this approach is to divide the control system design into two steps: 1) identification of a fuzzy model from input-output samples of the process [12], and 2) obtain a fuzzy controller from the fuzzy model.

The fuzzy model may consist of a set of fuzzy rules each of them specifying a linear equation (linear combination of process variables and control variables) in the consequent part of the rule. The antecedent parts of the rules define the regions (fuzzy regions) where the linear models are valid. The control systems designed using this approach are known as Takagi-Sugeno (TS) fuzzy control systems. Several studies have been done on the stability of these fuzzy systems [5, 13].

The paper is mainly devoted to stability studies for multiple inputs multiple outputs control loops as shown in Figure 1 where $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. In many stability studies it is assumed that the set points r_i $(i = 1...p)$ and the external perturbations are constant. In this case the system is said to be autonomous and the problem is to maintain stability in spite of perturbations that lead the system to different conditions.

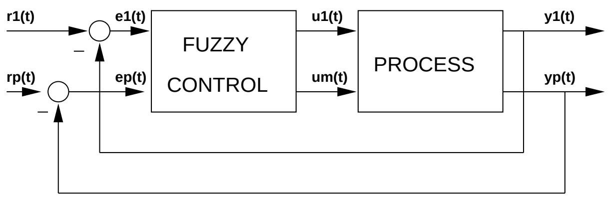

Figure 1: Multivariable fuzzy feedback control.

The multivariable fuzzy controller in Fig. 1 could be a Mamdani controller, with the error signals $e_i(i = 1...p)$ as inputs, and the control actions u_i $(i = 1...m)$ as outputs. Furthermore, it is possible to have a TS controller.

The next section introduces some concepts related to the stability methods. Section 3 presents a survey of existing input-output methods, frequency domain approaches, and Lyapunov methods for fuzzy systems. Finally, Section 4 is for the conclusions and perspectives.

2 STABILITY CONCEPTS AND METHODS

Several classifications of existing methods for stability analysis are possible. The most conventional one in the general stability theory is to distinguish between input-output methods and internal description or state variable methods.

Input-output (I/O) methods involves only relations between the external inputs and the outputs. A well known associated stability concept is the bounded input-bounded output or BIBO stability.

The basic assumption is that in well behaved systems bounded input should results in bounded outputs, and that small changes in inputs should result in small changes in outputs. In the input-output methods the systems can be represented by operators G which produce the output $y(t)$ corresponding to the input $u(t)$, or by relations G formed by all the pairs $(u(t), y(t))$ where $y(t)$ is a possible output produced by the input $u(t)$. There are several input-output stability methods. Some significant results are the small gain theorem, the conicity criterion [11], which is a generalization of the circle criterion for the multivariable case, and the passivity theorem.

Frequency response methods can also be considered as input-output stability methods even if the methodology is different. The multivariable generalization of the traditional frequency response methods include direct/inverse Nyquist, Mees bands, Gershgorin bands, and Characteristic Loci.

Internal description methods are related to properties of the internal representation of the system by means of state variables and the effect of perturbations resulting in changes of initial conditions.

Let $x \in \mathbb{R}^n$ be the state of the system. The dynamical system is described by the differential equation:

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. This function f describes the dynamic of the system and can be represented by a classical algebraic equation and/or a fuzzy knowledge based system (or a combination of both of them). It should be noted that (1) could represent the closed loops system in Fig.1 by assuming $r_1 = ... = r_p = 0$.

The most well known stability approach is the Lyapunov theory. There exist numerous definitions of Lyapunov stability. It is assumed that the system (1) is time-invariant, autonomous and admits a finite-dimensional state-space representation. All these assumptions are not required in some of the input-output approaches. Necessarily, $f(0) = 0$.

Then, the equilibrium 0 is said globally asymptotically stable with respect the system (1) if

$$\forall x_0 \in \mathbb{R}^n, \qquad x(t) \to 0 \qquad as \qquad t \to \infty \tag{2}$$

along all the trajectories of the system (1). Otherwise, the set of x_0 that satisfy (2) give rise to a bounded set (the attraction basin of the equilibrium point) and the stability of the equilibria is only local.

Thus, the asymptotic stability of the equilibrium point 0 in a Lyapunov sense is relative to an initial internal perturbation on the state vector and requires the state of the system to converge to the only existing equilibrium point 0. On the contrary, the I/O stability is relative to an external perturbation and requires the output of the system to be small enough with respect to the perturbation.

The Lyapunov methods are based on the determination of Lyapunov functions.

Qualitative methods also study the dynamic evolution of the system represented by (1) using the qualitative theory of nonlinear dynamical system. Stability and robustness indices were formulated to analyze conventional fuzzy control systems with rules defining linguistic control actions obtained from the process heuristic knowledge. The indices quantify how far the system is from loss of stability [1].

3 STABILITY ANALYSIS OF FUZZY SYSTEMS

3.1 INPUT-OUTPUT METHODS

If G is a linear time-invariant representation in the working point of the process to be controlled, its gain can be easily computed using its frequency response:

$$g(G) = \sup_{\omega} \bar{\sigma}(G(j\omega)) \tag{3}$$

where $\bar{\sigma}$ is the maximum singular value of the matrix $G(j\omega)$. Furthermore, if H is a nonlinear static fuzzy controller $H = \phi(e)$ the gain can be obtained as:

$$g(\phi) = \sup_{|e|\neq 0} \left\{ \frac{|\phi(e)|}{|e|} \right\} \tag{4}$$

The small gain theorem states that a sufficient condition for the stability of the closed-loop system in Fig. 1 is that $g(G)g(H) < 1$. That is the product of the gains should be lower than 1.

Several stability criteria can be stated from the applications of the above concepts. Particularly, the circle criterion has been proposed in [9] for the stability analysis of both single-input single-output (SISO) and multi-input multi-output (MIMO) Mamdani type fuzzy control system when the model of the plant is known and linear time invariant. That is a restrictive assumption from the point of view of the practical interest.

The conicity criterion can be used to generalize the above results. This criterion leads to the following sufficient conditions for the stability of the fuzzy control system [7]: $i_c < 1$ where i_c is called the conicity index which is defined as $i_c = r_h/r_g$, where $r_h(C) = g(H - C)$ is the conic deviation and $r_g(C) = 1/g(G(I+CG)^{-1})$ is the conic robustness. In the above expressions C is called centre of the cone. Thus, stability analysis reduces to find a cone centre such that $i_c < 1$.

3.2 FREQUENCY RESPONSE METHODS

The existing frequency response techniques for the analysis of fuzzy control systems have been traditionally applied to systems with a known linear model of the process and a nonlinear feedback (fuzzy controller). This model requirement was a significant limitation for practical usage in fuzzy control engineering. However, in FAMIMO a transformation technique has been developed, making possible to study stability of TS systems with these methods.

In FAMIMO the harmonic balance equation has been used to search for limit cycles. In this case this equation leads to

$$G(j\omega)N(a)y = -y, \tag{5}$$

where $y_i = a_i e^{j(\omega t + \theta i)}$ are the complex representation of sinusoids, and $N(a)$ is the describing function of the FLC. For a limit cycle to exist (5) must have a non trivial solution. To solve that equation, in the case where the nonlinearity is additively decomposable a method suggested by Mees [6] can be used. For a square G this method is based on the fact that (5) can only have a solution if $G^{-1}(j\omega)+N(a)$ has at least one zero eigenvalue. To check whether that happens the Gershgorin theorem is employed [10]. This theorem leads to the stability condition:

$$|\hat{g}_{kk}(j\omega) + n_{kk}| > \sum_{i\neq k}^{n} |\hat{g}_{ik}(j\omega)| + \sum_{i\neq k}^{n} |n_{ik}| \tag{6}$$

for $k = 1, .., n$. A different method is based on the direct analysis of the harmonic balance equation. Thus, the number of encirclements of the characteristic loci of $G(j\omega)N(a)$ around the point $(-1, 0)$ is studied [8].

A third method also developed in FAMIMO is based on the robust analysis of limit cycles using singular values. The method is applied to a system with a multiplicative error Δ model. A theorem to assure the absence of limit cycles has been presented [3]. This theorem defines the following condition on the model error:

$$\bar{\sigma}(\Delta(j\omega)) < \frac{1}{\bar{\sigma}(G(j\omega)N(a)(I + G(j\omega)N(a_0))^{-1})} \forall\omega, \forall a. \tag{7}$$

3.3 LYAPUNOV METHODS

The Lyapunov approach is today the most extended method to analyze the stability of TS fuzzy control systems. A continuous time linear TS fuzzy model is equivalent to the system:

$$\dot{x} = \left(\sum_{i=1}^{M} w_i(x) A_i \right) x + \left(\sum_{i=1}^{M} w_i(x) B_i \right) u \tag{8}$$

System (8) satisfies the convexity conditions and can be considered as a polytopic linear differential inclusion (PLDI), and written as

$$\dot{x} = A(t)x+B(t)u, [A(t)B(t)] \in Co\{[A_1B_1], ..., [A_nB_n]\} \tag{9}$$

i.e., $[A(t)B(t)]$ lies in the convex hull spanned by the matrices $[A_1B_1], ..., [A_nB_n]$.

Stability analysis of PLDI and, consequently, of linear TS systems [13, 14], is reduced to find a common matrix P valid for a set of linear matrix inequalities on the form $P > 0$ and $PA_i + A_i^T P < 0$, that can be solved by means of new convex optimization techniques [2]. The quadratic Lyapunov function is given by $V(x) = x^T Px$.

The Lyapunov approach is usually very conservative. Thus, it is not possible to find quadratic Lyapunov functions of many stable fuzzy control systems.

However, in case of membership functions with local support, which induce a partition in the space of the input variables, piecewise quadratic Lyapunov functions [4] offer many solutions to stability problems that cannot be solved with conventional techniques. A matrix P_i has to be computed for every region and the Lyapunov function is defined by $V(x) = x^T P_i x$.

4 CONCLUSIONS AND PERSPECTIVES

Stability analysis of fuzzy control systems is a topic that has attracted the attention of an increasing number of researchers in the control theory and mathematical system theory domains in the last ten years. This topic has a great practical interest in many control engineering applications in which safety and reliability are critical issues and where stability testing using only experimentation will not suffice to guarantee the stability property in all possible working conditions.

Some concepts related to stability have been reviewed. Several stability measures are based on the assumption that a precise mathematical model of the system under consideration is known. In practice, such a mathematical model is often unknown or its determination is too expensive. However, new methods combining modelling techniques from input-output data and expert rule information could provide the models required for the stability analysis. This approach seems to be very promising for many applications. That leads to the analysis of a TS model with a fuzzy logic controller in the control loop. Stability of the resulting structure can analyzed by means of the input-output approaches and frequency response techniques. Lyapunov methods could also be applied.

It should be noted that there is no a singled method better than the others. In fact several methods are complementary. Thus, some methods do not lead to any solution in some problems where others could guarantee the stability, but these could be very hard to apply due to the required information or data about the process.

Thus, even if some new techniques and computational tools are being produced, more research efforts are needed to produce new practical method for stability analysis and design of these systems.

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