ON NORMALISED FUZZY SYSTEMS FOR FUZZY CONTROL

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Summary

The main goal of this work is to formalize the use of fuzzy systems represented by Takagi-Sugeno form, so that the use of membership functions is well understood when designing the rules. It also aimed to apply these ideas later when designing fuzzy controllers, where dynamic models are used. First, static models are introduced. Second, some useful definitions are presented. Third, the main properties of normalized fuzzy systems are shown. Fourth, we give examples of application in which we use these fuzzy systems for function approximation. Fifth, we extend the results to dynamic systems. We finish with an improvement of Takagi-Sugeno fuzzy model that allows to obtain universal approximators for dynamic systems.

Keywords: T-S model, Fuzzy control, Dynamic models, Universal approximators.

1 STATIC FUZZY SYSTEMS

We will consider systems with n input variables $x_1, x_2, \ldots, x_n$, and one output y (in the case of multiple outputs, the superposition principle may be applied). If each input variable x_l, is given r_l fuzzy values $X_l^1, X_l^2, \ldots, X_l^{r_l}$, then the fuzzy system contains $r_1 r_2 \ldots r_n$ rules, each one of the form:

$$R^{i_1 \cdots i_n} : IF(x_1 is X_1^{i_1}) AND \ldots AND (x_n is X_n^{i_n})$$
$$THEN \hat{y}^{i_1 \cdots i_n}(x_1, \ldots, x_n) = y^{i_1 \cdots i_n}$$

with $i_1 \in \{1, \ldots, r_1\}$, ..., $i_n \in \{1, \ldots, r_n\}$.

If the input array $\mathbf{x}$ is defined as $\mathbf{x} = [x_1, \ldots, x_n]^T$, then the following compact notation may be used:

$$R^{i_1 \cdots i_n} : IF(\mathbf{x} is \mathbf{X}^{i_1 \cdots i_n}) THEN \hat{y}^{i_1 \cdots i_n}(\mathbf{x}) = y^{i_1 \cdots i_n}$$

with $\mathbf{X}^{i_1 \cdots i_n} = \left[X_1^{i_1} \ldots X_n^{i_n}\right]^T$ and $\left(\mathbf{x} is \mathbf{X}^{i_1 \cdots i_n}\right) = AND_{l=1}^{n}\left(x_l is X_l^{i_l}\right)$. Attention may be paid to the fact that, in general, $\hat{y}^{i_1 \cdots i_n}(\mathbf{x})$ will be a function of $\mathbf{x}$, except in the static case that is being considered now.

Finally, the output of the set of rules is obtained as:

$$y(\mathbf{x}) = \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \cdots i_n}(\mathbf{x}) \hat{y}^{i_1 \cdots i_n}(\mathbf{x})}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \cdots i_n}(\mathbf{x})}$$

so

$$y(\mathbf{x}) = \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \cdots i_n}(\mathbf{x}) y^{i_1 \cdots i_n}}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \cdots i_n}(\mathbf{x})}$$

where $w^{i_1 \cdots i_n}(\mathbf{x})$ is the weight of the rule $(i_1 \ldots i_n)^{th}$, for a specific value of $\mathbf{x}$. If we define $\mu_{X_l^{i_l}}(x_l)$ as the membership function associated to the fuzzy set $X_l^{i_l}$, then the previous weight may be calculated as

$$w^{i_1 \cdots i_n}(\mathbf{x}) = \prod_{l=1}^{n} \mu_{X_l^{i_l}}(x_l)$$

with $0 \le \mu_{X_l^{i_l}}(x_l) \le 1, \quad \forall x_l, \quad \forall l \in \{1, \ldots, n\}$.

2 DEFINITIONS

Definition 1 *A fuzzy system is considered normal, and is denoted by Normalized Fuzzy System (NFS), when* $\forall l \in \{1, \ldots, n\}$,

$$\sum_{i_l=1}^{r_l} \mu_{X_l^{i_l}}(x_l) = 1, \quad \forall x_l$$

$$and \mu_{X_l^{i_l}}(x_l) \ is\ a\ convex\ function$$

We also define:

$$x_{li_l}^- = min\{x_l | \mu_{X_l^{i_l}}(x_l) = 1\}, \quad \forall i_l \in \{2, \ldots, r_l\}$$

$$x_{li_l}^+ = max\{x_l | \mu_{X_l^{i_l}}(x_l) = 1\}, \quad \forall i_l \in \{1, \ldots, r_l - 1\}$$

$$x_{l1} = max\{x_l | \mu_{X_l^1}(x_l) = 1\}$$

$$x_{lr_l} = min\{x_l | \mu_{X_l^{r_l}}(x_l) = 1\}$$

being $X_l = [x_{l1}, x_{lr_l}]$ *the universe of discourse.*

If $x_{li_l}^- = x_{li_l}^+, x_{li_l} = \{x_l | \mu_{X_l^{i_l}}(x_l) = 1\} = x_{li_l}^- = x_{li_l}^+, \quad \forall i_l \in \{2, \ldots, r_{l-1}\}$.

Definition 2 *The NFS is monotonic if* $\forall i_l \in \{2, \ldots, r_l - 1\}$, *only exists one* $x_l | \mu_{X_l^{i_l}}(x_l) = 1$, *this means* $x_{li_l}^- = x_{li_l}^+ = x_{li_l}$

Definition 3 *The NFS is*

- *static if* $\hat{y}^{i_1 \ldots i_n}(\mathbf{x}) = y^{i_1 \ldots i_n}, \quad \forall i_1, \ldots, i_n$
- *functional if* $\hat{y}^{i_1 \ldots i_n}(\mathbf{x}) = f^{i_1 \ldots i_n}(\mathbf{x}), \quad \forall i_1, \ldots, i_n$
- *linear if* $\hat{y}^{i_1 \ldots i_n}(\mathbf{x}) = \sum_{l=1}^n a_l^{i_1 \ldots i_n} x_l + a_0^{i_1 \ldots i_n}$, $\forall i_1, \ldots, i_n$

Definition 4 *The points* $\left(\mathbf{x}^{i_1 \ldots i_n}, \ y^{i_1 \ldots i_n}\right) \in Re^{n+1}$, *with* $\mathbf{x}^{i_1 \ldots i_n} = [x_{1i_1}, \ldots, x_{ni_n}]^T$, *are called guided points when the NFS is monotonic, or the points obtained after replacing* x_{li_l} *by* $x_{li_l}^-$ *or* $x_{li_l}^+$ *if it is not monotonic. In general, the notation for both cases will be* $\left(\mathbf{x}^{-+i_1 \ldots i_n}, y^{i_1 \ldots i_n}\right)$.

These definitions are useful to present the following properties, that may be applied to most usual fuzzy systems.

3 PROPERTIES OF NFS

A NFS verifies the following properties, although their demonstration is not given due to the limited size allowed. Some of them may be found in [3].

Property 1 *The only membership function that has value in the points* $x_l = x_{li_l}^{+-}, \quad \forall i_l \in \{1, \ldots, r_l\}, \forall l \in \{1, \ldots n\}$ *is:*

$$\mu_{X_l^{j_l}}(x_{li_l}^{+-}) = \delta_{i_l j_l}$$

Property 2 $\forall x_{li_l}^+ < x_{li_l} < x_{li_l}^-$, *only two membership functions take value:* $\mu_{X_l^{i_l}}(x_l)$ *and* $\mu_{X_l^{i_l+1}}(x_l)$. *Furthermore,* $\mu_{X_l^{i_l+1}}(x_l) = 1 - \mu_{X_l^{i_l}}(x_l)$.

Property 3 $\forall x_l \notin X_l$, *this means, out of the universe of discourse, the only membership function that has value is* $\mu_{X_l^1}(x_l)$ *or* $\mu_{X_l^{r_l}}(x_l)$.

Property 4 $\sum_{i_1=1}^{r_1} \ldots \sum_{i_n=1}^{r_n} w^{i_1 \ldots i_n}(\mathbf{x}) = 1, \quad \forall \mathbf{x}$ *(see [4]).*

Property 5 *The output of the fuzzy system,* $y(\mathbf{x})$ *passes through the guide points.*

Property 6 *In non monotonic NFS,* $y(\mathbf{x}) = y^{i_1 \ldots i_n}$ *(constant), if* $x_l \in [x_{li_l}^-, x_{li_l}^+], \quad \forall l \in \{1, \ldots, n\}$.

Property 7 $y(\mathbf{x})$ *is constant outside the universe of discourse.*

4 FUNCTION APPROXIMATION WITH NFS

Lemma 1 *If it is possible to formulate* $y(\mathbf{x})$ *as* $y(\mathbf{x}) = f^{i_1 \ldots i_n}(\mathbf{x}) = g_1(x_1) \ldots g_n(x_n), \quad \forall x_l \in [x_{li_l}, x_{li_{l+1}}], \quad \forall l \in \{1, \ldots, n-1\}$, *then it is possible to approximate it with null error by using a static NFS.*

The main conclusion is that the membership functions of the NFS must be selected as follows:

$$\mu_{X_l^{i_l+1}}(x_l) = \frac{g_l^{i_l}(x_l) - g_l^{i_l}(x_{li_l})}{g_l^{i_l}(x_{li_{l+1}}) - g_l^{i_l}(x_{li_l})}$$

$$\mu_{X_l^{i_l}}(x_l) = 1 - \mu_{X_l^{i_l+1}}(x_l) = \frac{g_l^{i_l}(x_{li_{l+1}}) - g_l^{i_l}(x_l)}{g_l^{i_l}(x_{li_{l+1}}) - g_l^{i_l}(x_{li_l})}$$

It may be verified that the function $g_1^1(x_1) = -3.5x_1^2 + 2.5x_1^3$, which is not monotonic in $0 \le x_1 \le 1$ will lead to non monotonic membership functions:

$$\mu_{X_1^2}(x_1) = \frac{g_1^1(x_1) - g_1^1(0)}{g_1^1(1) - g_1^1(0)} = 3.5x_1^2 - 2.5x_1^3 \notin [0, 1]$$

$$\mu_{X_1^1}(x_1) = 1 - 3.5x_1^2 + 2.5x_1^3 \notin [0, 1]$$

The solution to this problem would be by decomposing the interval in two parts, taking as new point the one in which $\dot{y}(x_1) = 0$, this means $-7x_1 + 7.5x_1^2 = 0$, so $x_1 = 0.93$, and $y(0.93) = -1.016$:

For $0 \le x_1 \le 0.93$, we have

$$\mu_{X_1^2}(x_1) = \frac{g_1^1(x_1)-g_1^1(0)}{g_1^1(0.93)-g_1^1(0)} = 3.44x_1^2 - 2.46x_1^3 \in [0,1]$$

$$\mu_{X_1^1}(x_1) = 1 - 3.44x_1^2 + 2.46x_1^3 \in [0,1]$$

For $0.93 \leq x_1 \leq 1$, we have

$$\mu_{X_1^3}(x_1) = \frac{g_1^2(x_1)-g_1^2(0.93)}{g_1^2(1)-g_1^2(0.93)} = -218.75x_1^2 + 156.25x_1^3 + 63.5 \in [0,1]$$

$$\mu_{X_1^2}(x_1) = 1 + 218.75x_1^2 - 156.25x_1^3 - 63.5 \in [0,1]$$

Two examples of function approximation are given.

Example 1 $y(x_1) = x_1|x_1|, \quad \forall x_1 \in [-1,1]$

$$R^1 : IF(x_1 is X_1^1) THEN \hat{y}^1(x_1) = -1$$
$$R^2 : IF(x_1 is X_1^2) THEN \hat{y}^2(x_1) = 0$$
$$R^3 : IF(x_1 is X_1^3) THEN \hat{y}^3(x_1) = 1$$

with

$$\mu_{X_1^1}(x_1) = x_1^2, \quad \forall x_1 \in [-1,0]$$
$$\mu_{X_1^2}(x_1) = 1 - x_1^2, \quad \forall x_1 \in [-1,1]$$
$$\mu_{X_1^3}(x_1) = x_1^2, \quad \forall x_1 \in [0,1]$$

Example 2 $y(\mathbf{x}) = (x_1^2+2)(3 - sinx_2), \quad \forall x_1 \in [1,2], \quad \forall x_2 \in [\pi/2, \pi]$

$$R^{11} : IF(x_1 is X_1^1) AND (x_2 is X_2^1) THEN \hat{y}^{11}(x_1,x_2) = 6$$
$$R^{12} : IF(x_1 is X_1^1) AND (x_2 is X_2^2) THEN \hat{y}^{12}(x_1,x_2) = 9$$
$$R^{21} : IF(x_1 is X_1^2) AND (x_2 is X_2^1) THEN \hat{y}^{21}(x_1,x_2) = 12$$
$$R^{22} : IF(x_1 is X_1^2) AND (x_2 is X_2^2) THEN \hat{y}^{22}(x_1,x_2) = 18$$

with

$$\mu_{X_1^1}(x_1) = \frac{4-x_1^2}{3}, \quad \forall x_1 \in [1,2]$$
$$\mu_{X_1^2}(x_1) = \frac{x_1^2-1}{3}, \quad \forall x_1 \in [1,2]$$
$$\mu_{X_2^1}(x_2) = sen(x_2), \quad \forall x_2 \in [\pi/2, \pi]$$
$$\mu_{X_2^2}(x_2) = 1 - sen(x_2), \quad \forall x_2 \in [\pi/2, \pi]$$

Corolary 1 *Lets suppose that* $x_l \in [x_l i_l, x_{li_{l+1}}], \quad \forall l \in \{1,\ldots,n\}$. *If* $g_l^{i_l} \quad \forall l \in \{1,\ldots,n\}$ *is monotonic in that region, then* $y(\mathbf{x})$ *is also monotonic. So, the limits of* $y(\mathbf{x})$ *in the previous region are its* 2^n *guide points.*

Corolary 2 *In a NFS with linear membership functions, if* $x_l \in [x_{li_l}, x_{li_{l+1}}], \quad \forall l \in \{1,\ldots,n\}$, *then* $y(\mathbf{x})$ *is n-linear with respect to* $\mu_{X_l^{i_l}}(x_l), \quad \forall l \in \{1,\ldots,n\}$. *In a functional NFS,* $y(\mathbf{x})$ *has the same shape of the membership functions.*

These corolaries help to understand the function evolution between the guide points, and will be important when developing stability theorems.

5 DYNAMIC FUZZY SYSTEMS

The function approximation method shown in this work is similar to those given by other authors [1], [2], [6]. But the main contribution of this work is twofold: to formalize it with a proper notation and making use of what we call “normalised fuzzy systems”, and to extend it to the approximation of dynamic fuzzy systems, by improving Takagi-Sugeno fuzzy model [5].

Takagi-Sugeno (T-S) model corresponds to a functional NFS. A particular case, widely used due to its application to fuzzy control, is the case in which it becomes a linear NFS:

$$R^{i_1\ldots i_n} : IF(x_1 is X_1^{i_1}) AND \ldots AND (x_n is X_n^{i_n})$$
$$THEN \hat{y}^{i_1\ldots i_n}(\mathbf{x}) = \sum_{l=1}^{n} a_l^{i_1\ldots i_n} x_l + a_0^{i_1\ldots i_n}, \quad \forall i_1,\ldots,i_n$$

The model becomes automatically dynamic just taking $\mathbf{x}^T = [x_1,\ldots,x_n] = [x, \dot{x}, \ldots, x^{(n-1)}]$ and $y = x^n$. So,

$$R^{i_1\ldots i_n} : IF(x is X_1^{i_1}) AND \ldots AND (x^{(n-1)} is X_n^{i_n})$$
$$THEN x^{(n)} = \sum_{l=1}^{n} a_l^{i_1\ldots i_n} x^{(l-1)} + a_0^{i_1\ldots i_n}, \quad \forall i_1,\ldots,i_n$$

We will show now with an example that it is not possible to obtain a perfect approximator for the dynamic model.

Example 3 $\dot{x} = x|x|, \quad \forall x_1 \in [-1,1]$

$$R^1 : IF(x is X_1^1) THEN \dot{x} = 2x+1$$
$$R^2 : IF(x is X_1^2) THEN \dot{x} = 0$$
$$R^3 : IF(x is X_1^3) THEN \dot{x} = 2x-1$$

so $\dot{x} = x|x|$ for $x = \{-1, 0, 1\}$ (guide points), and $\frac{d\dot{x}}{dx} = \{-2x, 0, 2x\}$ for $x = \{-1, 0, 1\}$. This means that we are approximating both the funcion model and its derivative in the guide points. Nevertheless when we try to apply the previous method for $x \in [-1, 0]$ we obtain $\mu_{X_1^1}(x) = \frac{-x^2}{2x+1}$, which has no sense for a membership function in $x = -0.5$. The same happens for $x \in [0, 1]$ We could choose, for example $\mu_{X_1^1}(x) = -x$ (i.e. triangular membership functions), but we would obtain an approximation error.

6 LINEAR SYSTEMS WITH FUZZY DINAMICS

We propose an improvement in T-S model that allows to apply static NFS approximators to dynamic fuzzy system. The idea consists on decomposing the fuzzy system in $n+1$ fuzzy sub-systems, one for each linear sub-system coefficients. For a_l,

$$R^{i_1 \ldots i_n} : IF(xisX_{1l}^{i_1})AND \ldots AND(x^{(n-1)}isX_{nl}^{i_n})$$

$$THEN \hat{a}_l^{i_1 \ldots i_n} = a_l^{i_1 \ldots i_n}, \quad \forall i_1, \ldots, i_n$$

with different membership functios for each one. This allows to obtain a perfect approximation, by taking:

$$a_l(\mathbf{x}) = \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \ldots i_n}(\mathbf{x}) \hat{a}_l^{i_1 \ldots i_n}}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{i_1 \ldots i_n}(\mathbf{x})}$$

and $x^{(n)} = \sum_{l=1}^{n} a_l(\mathbf{x}) x^{(l-1)} + a_0(\mathbf{x})$

Example 4 $\dot{x} = x|x|, \quad \forall x_1 \in [-1, 1]$. *We will use two static NFS. For a_0,*

$$R^1 : IF(xisX_{11}^1)THEN \hat{a}_0^1 = 1$$

$$R^2 : IF(xisX_{11}^2)THEN \hat{a}_0^2 = 0$$

$$R^3 : IF(xisX_{11}^3)THEN \hat{a}_0^3 = -1$$

$$\mu_{X_{11}^1}(x) = x^2, \quad \forall x \in [-1, 0]$$

$$\mu_{X_{11}^2}(x) = 1 - x^2, \quad \forall x \in [-1, 1]$$

$$\mu_{X_{11}^3}(x) = x^2, \quad \forall x \in [0, 1]$$

For a_1,

$$R^1 : IF(xisX_{11}^1)THEN \hat{a}_1^1 = 2$$

$$R^2 : IF(xisX_{11}^2)THEN \hat{a}_1^2 = 0$$

$$R^3 : IF(xisX_{11}^3)THEN \hat{a}_1^3 = 2$$

with

$$\mu_{X_{12}^1}(x) = -x, \quad \forall x \in [-1, 0]$$

$$\mu_{X_{12}^2}(x) = x + 1, \quad \forall x \in [-1, 1]$$

$$\mu_{X_{12}^3}(x) = x, \quad \forall x \in [0, 1]$$

Now the approximator $\dot{x} = a_1(x)x + a_0(x)$ allows null error with the model.

7 CONCLUSION

The main result of this work is that a Normalized Fuzzy System allows to draw easily the relation $y(\mathbf{x})$, from the shape of the membership function. In fact, the use of NFS does not imply any drastical restriction, because they are widely used (indeed without knowing their properties), and they show the best performance (see [3]).

Then, the method has been extended to T-S fuzzy model, allowing the approximation of dynamic systems, which have their main use in fuzzy control. The continuation of this work will be to extend this improved T-S model to carry out stabilty analysis.

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