# FUZZY DEDUCTIVE REASONING AND ANALOGICAL SCHEME


**B. Bouchon-Meunier**
LIP6, UPMC, case 169,
4 place Jussieu
75252 Paris Cédex 05
France
Bernadette.Bouchon-Meunier@lip6.fr

**J. Delechamp**
LCPC
58 boulevard Lefebvre
75015 Paris
France
delecham@inrets.fr

**C. Marsala**
LIP6, UPMC, case 169,
4 place Jussieu
75252 Paris Cédex 05
France
Christophe.Marsala@lip6.fr

**R. Mesiar**
Dep. Mathem., FCE STU
Radilinského 11
813 68 Bratislava
Slovakia
mesiar@vox.svf.stuba.sk

**M. Rifqi**
LIP6, UPMC, case 169,
4 place Jussieu
75252 Paris Cédex 05
France
Maria.Rifqi@lip6.fr



## Summary

We propose to regard fuzzy deductive reasoning from the point of view of analogy and we merge it into the analogical scheme we have introduced [ 1].



**Keywords**: Analogy, compositional rule of inference, conjunctors.


## 1 INTRODUCTION

Analogical reasoning has been formalized in a fuzzy set based approach in various directions [3][4][7] and it seems natural that there is some relationships between approximate reasoning and analogical reasoning, since both of them appear as a modeling of human natural ways of reasoning. We have introduced in [1] a general framework of analogy which covers several reasoning methods in fuzzy logic, including well-known forms of fuzzy deductive reasoning. It appears that we can consider a less constrained definition of fuzzy deductive reasoning than the usual one, preserving the main properties and still in adequation with the analogical scheme we have proposed. We present this view of fuzzy deductive reasoning here.

## 2 ANALOGICAL SCHEME

We consider two variables X and Y, defined on universes $\mathsf{X}$ and $\mathsf{Y}$. Let us denote by $F(\mathsf{X})$ and $F(\mathsf{Y})$ respective finite subsets of fuzzy sets $[0,1]^{\mathsf{X}}$ of $\mathsf{X}$ and $[0,1]^{\mathsf{Y}}$ of $\mathsf{Y}$.
For a given relation $\beta$ on $[0,1]^{\mathsf{X}} \times [0,1]^{\mathsf{Y}}$ and two binary relations R on $[0,1]^{\mathsf{X}} \times [0,1]^{\mathsf{X}}$ and S on $[0,1]^{\mathsf{Y}} \times [0,1]^{\mathsf{Y}}$, an *analogical scheme* is a function $\Re_{\beta RS}$ : $F(\mathsf{X}) \times F(\mathsf{Y}) \times [0,1]^{\mathsf{X}} \to [0,1]^{\mathsf{Y}}$ satisfying :

$\forall B \in F(\mathsf{X})$ and $\forall C \in F(\mathsf{Y})$ such that $B\beta C$,
(i) $C = \Re_{\beta RS}(B,C,B)$
(ii) $\forall B' \in [0,1]^{\mathsf{X}}$ such that $BRB'$, then $C' = \Re_{\beta RS}(B,C,B')$ satisfies ($B'\beta C'$ and $CSC'$)

We can interprete this scheme as follows, assuming that $\beta$, R and S are defined properly : if B and C are known to be linked by $\beta$, and if $B'$ resembles B, there exists $C'$ such that $B'$ and $C'$ are linked by $\beta$ and $C'$ resembles C.

## 3 DEDUCTIVE REASONING

Let us study a population E of objects, each of them characterized by a value B of the variable X and a value C of the variable Y. We consider a base of rules ($R_j$) of the form "if X is $B_j$ then Y is $C_j$", with $B_j$ in F(X) and $C_j$ in F(Y), $j \in J$, with $B_j$ and $C_j$ normalized fuzzy sets. The base of rules is supposed to describe objects of the population E. For some objects of E, the only values of X are known and we look for the values of Y attached to them. There exist various ways of determining this value of Y : inductive reasoning, case-based reasoning for instance. In the following, we focus on deductive reasoning.

A link $\beta$ between fuzzy descriptions of X and fuzzy descriptions of Y is defined by :
$B\beta C \Leftrightarrow$ i) or ii) holds with :
i) $\exists$ an object of E attached to a value of X equal to B and a value of Y equal to C,
ii) $\exists R_j$ such that $B = B_j$ and $C = C_j$.

Let us consider a *conjunctor* $T : [0,1]^2 \to [0,1]$ supposed to be left-continuous with neutral element 1, for instance a continuous t-norm, which can be regarded as a non-decreasing extension of the Boolean conjunction.

We manage any fuzzy rule $R_j$ by means of the set of values

$$I\big(B_j(x), C_j(y)\big)\ x \in X, y \in Y$$

provided by a *fuzzy implication* defined as a function

$$I : [0,1]^2 \to [0,1],$$

such that :

$$I(1,u) = u, \ \forall u \in [0,1],$$
$$I \le I_T,$$

where $I_T$ denotes the *residual operator* associated with the conjunctor T :

$$I_T(u,v) = \sup\{z \in [0,1] / T(u,z) \le v\}.$$

An example of fuzzy implication is $I_T$ itself.

Deductive reasoning is achieved through the classical *compositional rule of inference* : for any fuzzy characterization $B'$ of X, we deduce from rule $R_j$ a description of Y defined by :

$$\forall y \in Y \ \ C'(y) = \sup_{x \in X} T\Big(B'(x), I\big(B_j(x), C_j(y)\big)\Big)$$

Our purpose is to show that this form of deductive reasoning can be regarded as a particular case of the analogical scheme presented in section 2.

# 4 ANALOGICAL SCHEME FOR DEDUCTIVE REASONING

Let us suppose that we are given two mappings r and s respectively defined on $[0,1]^X$ and $[0,1]^Y$ and taking values in [0,1], such that $r(B,B) = 1$ and $s(C,C) = 1$ for any B in $[0,1]^X$ and C in $[0,1]^Y$. Relations R and S are defined from r and s in such a way that there exist two thresholds $\rho$ and $\sigma$ in [0,1] such that :

$$\forall B \ \ \forall B' \in [0,1]^X \ \ BRB' \Leftrightarrow r(B,B') \ge \rho$$
$$\forall C \ \ \forall C' \in [0,1]^Y \ \ CSC' \Leftrightarrow s(C,C') \ge \sigma .$$

This implies that we have

$$\forall B \in [0,1]^X \ \ BRB,$$
$$\forall C \in [0,1]^Y \ \ CSC.$$

The function $\Re_{\beta RS}$ is defined as follows :

$\Re_{\beta RS}(B,C,B') = C' \Leftrightarrow C'$ is obtained from B, C, $B'$ by means of the compositional rule of inference.

It can be proven that $\Re_{\beta RS}$ is an analogical scheme with convenient choices for the relations R and S.

### 4.1 CONJUNCTIVE RULES

We can first consider as an implication I the conjunctor T. Let r and s be defined as the following measures of similitude [2] :

$$\forall B \ \ \forall B' \in [0,1]^X \ \ r(B,B') = \sup_{x \in X} T(B'(x), B(x)),$$
$$\forall C \ \ \forall C' \in [0,1]^Y \ \ s(C,C') = \sup_{y \in Y} T(C'(y), C(y)) .$$

It can be proved that $C' = \Re_{\beta RS}(B,C,B')$ is such that : $s(C,C') = r(B,B')$ and, consequently, $BRB'$ implies $CSC'$ for any thresholds $\rho$ and $\sigma$, with $\sigma \le \rho$.

With these definitions, $\Re_{\beta RS}$ is an analogical scheme, as defined in section 2.

For instance, if we take T=min, we get [1] :

$$I(x,y) = \min\big(B(x), C(y)\big) \text{ (Mamdani)}$$
$$r(B,B') = \sup_{x \in X}\Big(\min\big(B(x), B'(x)\big)\Big),$$
$$s(C,C') = \sup_{y \in Y}\Big(\min\big(C(y), C'(y)\big)\Big).$$

If we take T as the product, we get [1] :

$$I(x,y) = B(x).C(y) \text{ (Larsen)}$$
$$r(B,B') = \sup_{x \in X}\big(B(x).B'(x)\big),$$
$$s(C,C') = \sup_{y \in Y}\big(C(y).C'(y)\big).$$

### 4.2 IMPLICATIVE RULES

Let us now consider a fuzzy implication I as defined in section 3 satisfying the two following properties :

$$I(a,b) = 1 \text{ if } a \le b,$$
$$I(a,0) = 0 \text{ if } a > 0.$$

For instance, I can be the residual operator of a t-norm T* without zero divisors.

Then, let us consider r and s defined as the following measures of satisfiability :

$$r(B,B') = 1 - \sup_{\{x \in X / B(x)=0\}} B'(x),$$
$$s(C,C') = 1 - \sup_{\{y \in Y / C(y)=0\}} C'(y).$$

With these definitions, $C' = \Re_{\beta RS}(B,C,B')$ is such that :

$$s(C,C') = r(B,B')$$

and, consequently, $BRB'$ implies $CSC'$ for any thresholds $\rho$ and $\sigma$, with $\sigma \le \rho$. $\Re_{\beta RS}$ is again an analogical scheme.

For instance, the following fuzzy implications can be chosen :

$$\text{(i)} \ \ I(x,y) = \begin{cases} 1 \text{ if } x \le y \\ 0 \text{ otherwise} \end{cases} \ \text{(Rescher – Gaines)}$$

T can be chosen arbitrarily and Ran B is dense in [0,1] (e.g., B is a triangular fuzzy number, L-R fuzzy number, exponential fuzzy number, etc).

$$\text{(ii)} \ \ I(x,y) = \begin{cases} 1 \text{ if } x \le y \\ C(y) \text{ otherwise} \end{cases} \ \text{(Brouwer – Godel}$$

T can be chosen arbitrarily.

$$\text{(iii)}\ \ I(x,y) = \begin{cases} \min\left(\frac{y}{x},1\right) & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{(Goguen)}$$

T is a left-continuous conjunctor bounded by the product.

Another choice of mappings is the following :

$$r(B,B') = \inf_{x \in X} I_T\big(B'(x), B(x)\big),$$

$$s(C,C') = \inf_{y \in Y} I_T\big(C'(y), C(y)\big).$$

Then for any fuzzy implication I defined in section 2, we get $C' = \Re_{\beta RS}(B,C,B')$ such that :

$$s(C,C') \geq r(B,B')$$

and, consequently, $BRB'$ implies $CSC'$ for any thresholds $\rho$ and $\sigma$, with $\sigma \leq \rho$. We deduce that $\Re_{\beta RS}$ is an analogical scheme.

For instance, if T is the Lukasiewicz t-norm, we have :

$$r(B,B') = \inf_{x \in X} \min\big(1 - B'(x) + B(x), 1\big),$$

$$s(C,C') = \inf_{y \in Y} \min\big(1 - C'(y) + C(y), 1\big),$$

and any of the well-known following fuzzy implications can be used [1] :

$$I(x,y) = 1 - B(x) + B(x)C(y) \ \text{(Reichenbach)}$$

$$I(x,y) = \max\big(1 - B(x), C(y)\big) \ \text{(Kleene} - \text{Dienes)}$$

$$I(x,y) = \min\big(1 - B(x) + C(y), 1\big) \ \text{(Lukasiewicz)}$$

Symmetric mappings r and s, see [6], which can be considered as measures of resemblance, can also be used together with a given fuzzy implication I, in such a way that the function $\Re_{\beta RS}$ is an analogical scheme. This will be true for instance with the following definition :

$$r(B,B') = \inf_{x \in X} \min\big(I_T(B(x), B'(x)), I_T(B'(x), B(x))\big),$$

$$s(C,C') = \inf_{y \in Y} \min\big(I_T(C(y), C'(y)), I_T(C'(y), C(y))\big).$$

For instance, taking T as the Lukasiewicz t-norm yields

$$r(B,B') = 1 - \sup_{x \in X} \left|B(x) - B'(x)\right|,$$

$$s(C,C') = 1 - \sup_{y \in Y} \left|C(y) - C'(y)\right|,$$

while choosing the product t-norm for T, we obtain :

$$r(B,B') = \exp\left(\sup_{x \in X} \left|\log B(x) - \log B'(x)\right|\right),$$

$$s(C,C') = \exp\left(\sup_{y \in Y} \left|\log C(y) - \log C'(y)\right|\right).$$

Other definitions of r and s, associated with fuzzy implications I, are under investigation.

## 5 CONCLUSION

We have presented an analogical view of fuzzy deductive reasoning, weakening the traditional requirements regarding definitions of conjunction and implication and we have proven that there exist measures of satisfiability or resemblance such that the function $\Re_{\beta RS}$ provided by the compositional rule of inference is an analogical scheme.

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