ON THE CONSTRUCTION OF M-TRANSITIVE RELATIONS.

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Abstract

The transitivity of fuzzy relations with respect to a subset of quasi-linear functions in the unit interval is studied. A new construction of transitive relations with respect to an archimedean t-norm is proposed.
Keywords: fuzzy relation, transitivity, quasi-inverse, fuzzy connective

1. INTRODUCTION[1]

All the branches of human knowledge have the need to distinguish the objects they are working with, therefore it is necessary to define some type of "equality". From a classical point of view, this equality is an equivalence relation defined on the objects of the universe of discourse. Nevertheless, an increasing number of problems arising mainly from applied sciences and specially from the so-called soft-sciences, point out at the introduction of equalities that are not transitive in the usual sense [11,10,12]. A new approach is needed. This new point of view is obtained by introducing the concept of "degree of equality or similarity". This setting leads to a definition of equality linked with the idea of "proximity" having a clear topological flavour. The first step is to generalise the concept of binary relation by introducing the fuzzy relations in the universe of discourse X. In this context the definition of the transitive property plays a key role [7, 6,14, 15]. A fuzzy relation R defined on a set X fulfils some kind of transitivity if, given three elements x, y, z of X it is possible to calculate a lower bound of the value $R(x, z)$ given the values of $R(x, y)$ and $R(y, z)$. In a general framework this calculus is carried out by means of binary operation $M{:}[0,1]\times[0,1] \to [0,1]$ that can be interpreted as a conjunction [3] in the multivalued logic underlying the fuzzy sets theory. Therefore it seems natural to suppose that is a commutative operation. Further, if we want to include in this schema the classical transitive relations, then $M(1,1)=1$. Summing up, a fuzzy relation R defined in X is M-transitive if there is a commutative operation $M{:}[0,1]\times[0,1] \to [0,1]$ such that $M(1,1)=1$ and

$$M(R(x, y), R(y, z)) \leq R(x, z)$$

Under these hypotheses, several authors [3,15 ,7,8,5] have published a large number of papers devoted to this type of relations taking the operation M within the set $\mathcal{T}$ of t-norms. In this paper we study some aspects of the relations that are transitive with respect to operations that are more general relations than the t-norms such as the quasi-linear functions [1].

2. PRELIMINARIES

Throughout this paper I will represent the unit interval [0, 1] and M a mapping $I\times I\to I$ fulfilling the following conditions:

2.1 $M(1, 1)=1$.
2.2 $M(x, y) = M(y, x)$ for all $x, y \in I$
2.3 M is non-decreasing with respect to each variable.
2.4 M is continuous.

Definition 2.1
A fuzzy relation R defined on a set X is M-transitive if

$$M(R(x, y), R(y, z)) \leq R(x, z),$$

for all x, y, z of X.

Definition 2.2
Given $(x, y)\in I \times I$, the *residuation* of (x, y) with respect to M is the number

$$\hat{M}(x \mid y) = Sup\,\{\alpha \in [0,1] \mid M(\alpha, x) \leq y\} \qquad (1),$$

if it exists.
Given $A\subset I\times I$, if for all $(x, y)\in A$ there exists $\hat{M}(x|\,y)$ then, we can define a new operation $\hat{M}: A\to[0,1]$ given by the formula (1).
This operation will be termed as the *quasi-inverse* of M in A.
All the preceding definitions are very general, from now on we will focus our attention to a particular set of M-operations, the subset of the quasi-linear functions [1] defined by

$$M(x, y) = f^{[-1]}(p\, f(x) + q\, f(y)) \qquad (2).$$

Where $p, q \in I$ and $f : I \to [0,+\infty]$ is a strictly monotonous function [1,2].

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The conditions 2.1-4 fulfilled by the operation M impose the following restrictions to p, q and f:

2.7 $M(1,1) = 1 \Rightarrow f(1) = 0$ or $f(1) = +\infty$.

2.8 M commutative $\Rightarrow p = q$.

2.9 If M is interpreted as a conjunction [4], then $M(x,y) \leq (x+y)/2$ for all $(x, y)\in I\times I \Rightarrow f$ is a strictly decreasing function with $f(1)=0$ and $p\geq 1/2$.

In the formula (2), $f^{[-1]}$ represents the pseudo-inverse of f in the usual sense. Nevertheless, in this paper it will be used in more general way.

Definition 2.3

Given an strictly decreasing function f:[0,1] $\rightarrow$ [0,+∞], the pseudo-inverse of f is the function

$$f^{[-1]}:[-\infty,+\infty]\rightarrow[0,1]$$

defined in de following way:

$$f^{[-1]}(a)=\begin{cases} Inf\ f^{-1}([0,a]) & a\geq 0 \\ 1 & a<0 \end{cases}$$

The pseudo-inverse defined in this way is always a continuous function and coincides with its unique quasi-verse [8], therefore $f^{[-1]}\circ f = id_{[0,1]}$ and $\left[f^{[-1]}\right]^{[-1]} = f$ if f is continuous.

Summing up, $M(x,y)= f^{[-1]}(p(f(x)+f(y)))$, where f is a strictly decreasing continuous function, $f^{[-1]}$ its pseudo-inverse and $1\geq p\geq 1/2$

The set of this type of functions will be denote by $\boldsymbol{M}$. Each function is determined by a unique f, except for constant factors, and a value $p\in I$.

The operation M is called *strict* or *non-strict* depending on whether $f(0)=+\infty$ or $f(0)=k<+\infty$ respectively. The set $\boldsymbol{M}$, even being restricted to aforementioned conditions, contains a vast number of operations including the Archimedean t-norms (p=1) and the quasi-arithmetic means (p=1/2). In particular, for $f(x)=-lnx$, $p=1/2$ we have $M(x,y)=\sqrt{xy}$ (geometric mean) and for $f(x)= 1-x$, $p=1/2$ we have $M(x,y)=\dfrac{x+y}{2}$ (arithmetic mean)

If $M\in\boldsymbol{M}$ is generated by f, the t-norm $T_f(x,y)= f^{[-1]}(f(x)+f(y))$ will be termed the t-norm associated to the family

$$M_f = \{M \mid M(x,y)= f^{[-1]}(p(f(x)+f(y)); p\in [1/2,1]\}$$

3. TRANSITIVITY AND RESIDUATION.

In this section we study the relation between M–transitive relations ($M\in\boldsymbol{M}_f$) and the fuzzy relations generated by using the residuation of M.

Proposition 3.1

Given $M\in\boldsymbol{M}_f$ *if for* $(x,y)\in I\times I$ there exists $\hat{M}(x\mid y)$ then

$$\hat{M}(x\mid y)= f^{[-1]}\left(\frac{f(y)}{p}-f(x)\right)$$

In order to study which operations of the set $\boldsymbol{M}$ admit a residuation we need the following lemmas.

Lemma 3.1

If $M_f\in\boldsymbol{M}$ is strict then for all $x\in[0,1]$,

$$[0,x]\subset Im\,M_f(\cdot,x)$$

Lemma 3.2

Let $M\in\boldsymbol{M}_f$ be a non-strict function with $f(0)=k$. If $p\geq 1/2$ then, for all $x\in\left[0, f^{-1}\left(\left(\frac{1-p}{p}\right)k\right)\right]$ we have $[0,x]\subset Im\,M(.,x)$

From the preceding lemmas it immediately follows.

Theorem 3.1

If $M\in\boldsymbol{M}_f$ is strict, then M has a quasi-inverse $\hat{M}$ in the unit interval.

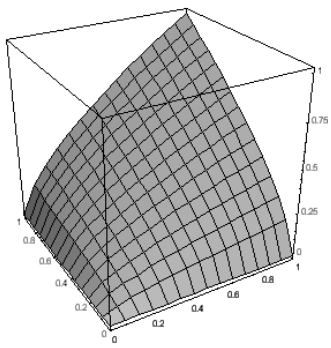

Figure 1

Example 3.1

If $f(x)=-\ln(x)$ then $f^{[-1]}(x)= Max(e^{-x},1)$, $M(x,y)=\sqrt[p]{x\cdot y}$ (figure 1) and

$$\hat{M}(x\mid y) = Min\left(\frac{y^{\frac{1}{p}}}{x},1\right) \quad (p=2/3).$$ Figure 2.

Theorem 3.2

Given a non-strict $M_f \in\boldsymbol{M}$ with $f(0)=k$, $p>1/2$ and $\lambda = f^{-1}\left(\frac{1-p}{p}k\right)$, M_f has a quasi-inverse in the domain A=[0,λ]× [0,λ]

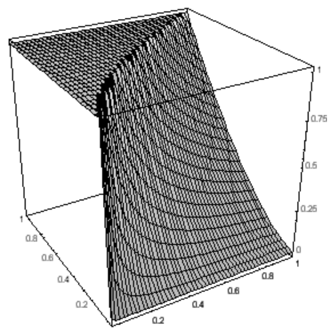

Figure 2.

Example 3.2

If $f(x) = 2(1-x)$ and $p=2/3$ then

$$f^{[-1]}(x)=\begin{cases} Min\left(1,1-\dfrac{x}{2}\right), & x\le 2\\ 0 & \text{otherwise}\end{cases}$$

and

$$M(x,y)= f^{[-1]}\left[\frac{2}{3}(2-2x+2-2y)\right]=$$

$$Max\left(\frac{2x+2y-1}{3},0\right) \qquad \text{(Figure 3)}$$

$$\hat{M}(x|y)=Min\left(1,\frac{1+3y-2x}{2}\right) \quad \text{for } x\in[0,1/2]$$

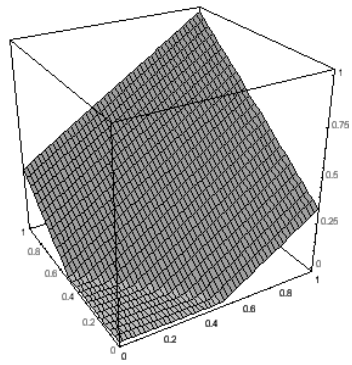

Figure 3

Remark 3.1

As it is shown in theorem 3.2, in the case of a non-strict operation with $p>1/2$, $\hat{M}(x\mid y)$ is not defined in the whole unit interval. In this case, we propose an extension of $\hat{M}(x\mid y)$ to the unit interval in the following way:

$$\hat{M}(x\mid y)= f^{[-1]}\left(\frac{f(y)}{p}-f(x)\right)$$

The function $\hat{M}(x\mid y)$ is well defined in the unit interval, it coincides with the residuation of M in the points where it exists and it has the value zero in the remaining points. Effectively, If $M(0,x)>y$ then

$$p(f(0)+f(x))<f(y)\Leftrightarrow \frac{f(y)}{p}-f(x)>f(0) \text{ and}$$

$$f^{[-1]}\left(\frac{f(y)}{p}-f(x)\right)=0$$

The generalised residuation for the example 3.2 is

$$\hat{M}(x|y)=Max\left(Min\left(1,\frac{1+3y-2x}{2}\right),0\right)$$

for $(x,y)\in I\times I$ (Figure 4)

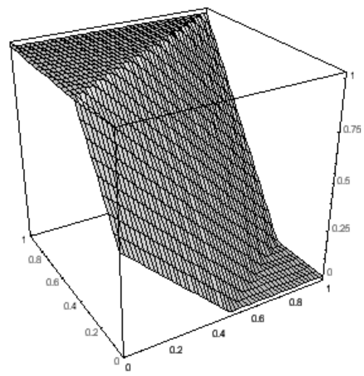

Figure 4: Generalised $\hat{M}(x\mid y)$

Let us denote by $\hat{\boldsymbol{M}}$ the subset of those operations of ***M*** that have residuation in the unit interval

Theorem 3.3

Let R be a M-transitive relation ($M\in\hat{\boldsymbol{M}}$) defined on a set X and $h_z=R(z,\cdot)$ $(z\in X)$, then

$$R(x,y)\le \inf_{z\in X} \hat{M}(h_z(x)\mid h_z(y)) \qquad (3)$$

The preceding result indicates that any M-transitive relation ($M\in\hat{\boldsymbol{M}}$) has the relation

$$R^*=\inf_{z\in X} R_M\circ(h_z\times h_z)$$

as an upper bound, being R_M the residuation of M considered as a fuzzy relation in I.

On the other hand, it is worth noting that, in general, the relation R_M is not M-transitive (except for $p=1$). Therefore, given a fuzzy set $h\in[0,1]^X$, the fuzzy relation R defined by $R(x.y)=R_M(h(x)|h(y))$ $(p\neq 1)$ is not M-transitive and neither it is the relation (3) of theorem 3.3. Nevertheless, we have the following result

Theorem 3.4

If $M_f \in \hat{\boldsymbol{M}}$, then the relation R_{M_f} is T_f transitive

Remark 3.2

In the previous proof of theorem 3.4, the definition 2.3 of quasi-inverse has not been used. The reasoning is based on the definition proposed in the remark 3.2. Therefore, theorems 3.3 and 3.4 are also valid by using this second one that it only affects to the non-strict operations.

Summarising the two preceding theorems we have the following result

Theorem 3.5

If R is a M-transitive relation ($M \in \boldsymbol{M}$) on a set X then

$$R \le R^* = \operatorname*{Inf}_{z \in X} R_M \circ (h_z \times h_z), \qquad (4)$$

where $h_z(x)=R(z,x)$ and R^* is a T_f -transitive relation (T_f is the archimedean t-norm associate to M).

This result gives us a method to generate non-reflexive relations that are transitive with respect to an arcimedean t-norm.

Theorem 3.6

Given a family of fuzzy sets $\{\mu_j\}_{j \in J}$ of X and $M \in \hat{\boldsymbol{M}}$, the fuzzy relation

$$R^*(x,y) = \operatorname*{Inf}_{j \in J} R_M (\mu_j(x) | \mu_j(y))$$

is a T_f-transitive relation, being T_f the t-norm associated to M.

Remark 3.3

If $M = T_f$, (p=1), then R_M is a reflexive relation. R^* is a preorder and $R=R^*$ (4) since it is a particular case of the representation theorem for similarity relations [15].

Finally, by symmetrising the relation R^*, we can generate symmetric and transitive relations with respect to an archimedean t-norm, that are not necessarily reflexive.

Theorem 3.7

Given a family of fuzzy sets $\{\mu_j\}_{j \in J}$ of X and $M \in \hat{\boldsymbol{M}}$, the fuzzy relation

$$R^*(x,y) = \operatorname*{Inf}_{j \in J} R_M \,(Max(\mu_j(x), \mu_j(y)) | \; Min(\mu_j(x), \mu_j(y)))$$

is a symmetric and T_f-transitive relation and not necessarily reflexive.

References

[1] Aczél J. (1966), Lectures on Functional Equations and their Applications, Academic Press, New York-London.

[2] Alsina,C. (1984), Between Min and Max. Proc. FISAL-84, UIB 37-40.

[3] Alsina C., Trillas E.; Valverde L., (1981), On some Logical Connectives for Fuzzy Sets Theory. JMMAA. 93, 15-26.

[4] Alsina, C., Cubillo, S., Jacas, J., Trillas, E. (1992), On And in [0,1]. Proceedings IPMU 92, Palma de Mallorca.

[5] De Baets, B, Mesiar R.(1998), T-partitions. Fuzzy Sets & Systems vol. 97-2 , 211-223.

[6] Fodor J.C. (1992) Traces of fuzzy binary relations. Fuzzy Sets & Systems vol. 50-3 , 331-341.

[7] Jacas, J.; Recasens, J., (1993), Fuzzy T-transitive relations: eigenvectors and generators, Fuzzy Sets and Systems, vol. 72, 2, 147-154.

[8] Klement E. P., Mesiar R., Pap E. (1999). Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. Fuzzy Sets & Systems vol. 104-1 , 3-13.

[9] Poncaré, H. (1943), La Ciencia y la Hipótesis. Espasa-Calpe.

[10] Menger, K., Probabilistic theory of relations. Proc. Nat. Acad. Sci. USA, 37, (1951), 178-180.Poincaré, H., (1943), La Ciencia y la Hipótesis, Espasa-Calpe.

[11] Trillas, E., (1982), Assaig sobre les relacions d'indistinguibilitat. Actes del Primer Congrés Català de Lògica Matemàtica, Barcelona, 51-54.

[12] Trillas, E.; Alsina, C., (1992), Logic: going further from Tarski, Fuzzy Sets and Systems, North Holland..

[13] Trillas, E.; Valverde, L., (1984), On implication an indistinguishability in the setting of fuzzy logic, in: J. Kacprzyk and R.R. Yager Eds., Management Decision Support Systems Using Fuzzy Sets and Possibility Theory (Verlag, TUV, Rheinland).

[14] Trillas, E.; Valverde, L., (1984), An inquiry into indistinguishability operators in: A. Skala et alt. eds., Aspects of Vagueness (Reidel Dordrecht, 1984), 231-256.

[15] Valverde, L, (1985), On the structure of F-indistinguishability operators. Fuzzy Sets and Systems, 17, 313-328.