DIVERGENCE MEASURES BASED ON T_∞

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Summary

In this paper we deal with divergence measures between two fuzzy sets when we have a partition of the data. We characterize this divergence when all data are equally important and we show the relation between divergence measures, fuzzy measures and pseudo metrics.

Palabras Clave: Fuzzy Partition, Divergence Measure, Fuzzy Measure.

1 INTRODUCTION AND BASIC CONCEPTS

The primary objective of clustering techniques is to partition a given data set into homogeneous clusters, and a practical problem that clustering methods must address is how to measure the similarity. In this sense, divergence measures [1], [2], [3], [4], [7], are used in order to quantify the difference between two fuzzy subsets of a universal set Ω. These measures try to maintain the properties of classical divergence measures between two probability distributions which appear in Information Theory [6],[9]. We can also use divergence measures to quantify the separation among clusters on a fuzzy partition, and our paper is a first step in the analysis of these results using divergence measures.

In the following, divergence measures are denoted as D, fuzzy subsets are denoted by A, B, Ω is the universal finite set, $|\Omega| = n$, and S_∞ fuzzy union and T_∞ intersection are considered, i.e. $(A \cup B)(x) = \min\{A(x) + B(x), 1\}$ and $(A \cap B)(x) = \max\{A(x) + B(x) - 1, 0\}$. We use these t-norm and t-conorm due to their relation with fuzzy partition [5].

Let us now give some definitions:

Definition 1 *Let Ω be the universal set and let $\widetilde{\mathcal{P}}(\Omega)$ be the set of all fuzzy subsets. An application $D : \widetilde{\mathcal{P}}(\Omega) \times \widetilde{\mathcal{P}}(\Omega) \to \mathbb{R}$ is said to be a* **divergence measure between fuzzy subsets** *if and only if the three following axioms hold:*

d1. $D(A, B) = D(B, A)$.

d2. $D(A, A) = 0$.

d3. $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$, $\forall C \in \widetilde{\mathcal{P}}(\Omega)$.

for any $A, B, C \in \widetilde{\mathcal{P}}(\Omega)$.

Definition 2 *Let D be a divergence measure. We say that D is* **local** *if and only if there exists a function $h : [0,1] \times [0,1] \to \mathbb{R}$ verifying*

$$D(A, B) - D(A \cup \Omega^i, B \cup \Omega^i) = h(A(x_i), B(x_i)), \quad \forall A, B \in \widetilde{\mathcal{P}}(\Omega), \quad \forall x_i \in \Omega$$

where

$$\Omega^i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

In fact, a local divergence measure says that each coordinate is independent of the others and that they are all equally important.

We can easily extend the definition of fuzzy measure to fuzzy subsets as:

Definition 3 *A* **fuzzy measure** *on $\widetilde{\mathcal{P}}(\Omega)$ is a set function $\mu : \widetilde{\mathcal{P}}(\Omega) \to [0,1]$ satisfying the following axioms:*

1. $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$.
2. $A \subset B \subset \Omega$ *implies* $\mu(A) \leq \mu(B)$.

Till the end, we will consider this definition.

2 DIVERGENCE MEASURES BASED ON T_∞

Proposition 1 *Let D be a divergence measure. Then $D(A,B) \geq 0$, $\forall A,\ B \in \widetilde{\mathcal{P}}(\Omega)$.*

Proof. From

$$D(A,B) \geq D(A\cap\emptyset, B\cap\emptyset) = D(\emptyset,\emptyset) = 0$$

■

Proposition 2 *Let A, B, C be fuzzy sets such that $A \subseteq B \subseteq C$. Then $D(A,B) \leq D(A,C)$ and $D(B,C) \leq D(A,C)$.*

Proof. To show that $D(A,B) \leq D(A,C)$, let H_1, H_2 be fuzzy sets such that $H_1(x) = (1 - C(x)) + (B(x) - A(x))$ and $H_2(x) = A(x)$, $\forall x \in \Omega$. H_1 is well defined since $A(x) \leq B(x) \leq C(x) \leq 1$.

$$\begin{aligned}
&(A\cap H_1)(x) = \\
&= \max\{ \quad A(x) + (1 - C(x)) + \\
&\qquad +(B(x) - A(x)) - 1, 0\} = 0 \\
&((A\cap H_1)\cup H_2)(x) = \\
&\qquad = \min\{0 + A(x), 1\} = A(x)
\end{aligned}$$

$$\begin{aligned}
&(C\cap H_1)(x) = \\
&= \max\{ \quad C(x) + (1 - C(x)) + \\
&\qquad +(B(x) - A(x)) - 1, 0\} \\
&= \qquad B(x) - A(x) \\
&((C\cap H_1)\cup H_2)(x) = \\
&= \min\{ \quad (B(x) - A(x)) + A(x), 1\} = B(x)
\end{aligned}$$

Thus

$$\begin{aligned}
&D(A,C) \geq \\
&\geq D(A\cap H_1, C\cap H_1) \geq \\
&\geq D((A\cap H_1)\cup H_2, (C\cap H_1)\cup H_2) = D(A,B)
\end{aligned}$$

To show that $D(B,C) \leq D(A,C)$, let be $H_3(x) = 1 - B(x)$, $H_4(x) = B(x)$ then

$$\begin{aligned}
&((A\cap H_3)\cup H_4)(x) = \\
&= \min\{ \quad \max\{A(x) + (1 - B(x)) - 1, 0\} + \\
&\qquad +B(x), 1\} \\
&= \min\{ \quad 0 + B(x), 1\} = B(x)
\end{aligned}$$

$$\begin{aligned}
&((C\cap H_3)\cup H_4)(x) = \\
&= \min\{ \quad \max\{C(x) + (1 - B(x)) - 1, 0\} + \\
&\qquad +B(x), 1\} \\
&= \min\{ \quad C(x) - B(x) + B(x), 1\} = C(x)
\end{aligned}$$

Then $D(B,C) \leq D(A,C)$. ■

Corollary 1 *If $A \subseteq B \subseteq C \subseteq H$ then $D(B,C) \leq D(A,H)$.*

Theorem 1 *If we have a finite universal set Ω, then D is a local divergence measure if and only if there exists an application $g : [0,1] \to R$ such that*

$$D(A,B) = \sum_{i=1}^{n} g(|A(x_i) - B(x_i)|),$$

verifying the following conditions:

- *$g(0) = 0$;*
- *$g(\cdot)$ is a monotonous non decreasing function in $[0,1]$.*

Proof. Let be $A_m(x) = (A \cup (\cup_{i=1}^m \{x_i\})(x)$, and $B_m(x) = (B \cup (\cup_{i=1}^m \{x_i\})(x)$, $m = 0,\ldots,n$, $\forall x \in \Omega$, thus $A_{m+1} = A_m \cup \{x_{m+1}\}$, and $A_n = \Omega = B_n$. If D is local, then

$$\begin{aligned}
&D(A_m, B_m) - D(A_m \cup \{x_{m+1}\}, B_m \cup \{x_{m+1}\}) = \\
&\quad = h(A(x_{m+1}), B(x_{m+1})),\ m = 0,\ldots,n-1.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{i=0}^{n-1} D(A_m, B_m) - \\
&\quad - \sum_{i=0}^{n-1} D(A_m \cup \{x_{m+1}\}, B_m \cup \{x_{m+1}\}) = \\
&\quad = D(A,B) - D(\Omega,\Omega) = D(A,B) \\
&\quad = \sum_{i=1}^{n} h(A(x_i), B(x_i))
\end{aligned}$$

Let us prove that $g(|a-b|) = h(a,b)$, $g(0) = 0$ and the monotony of g. Let a, b, $c \in [0,1]$, and $A(x) = a$, $B(x) = b$, $C(x) = c$, $\forall x \in \Omega$.

1. $h(a,a) = 0$, $\forall a \in [0,1]$. If there exists such $a \in [0,1] \mid h(a,a) \neq 0$, then $D(A,A) = n \cdot h(a,a) \neq 0$. Absurd.
2. $h(a,b) = h(b,a)$. If there exist a, b such that $h(a,b) \neq h(b,a)$, thus $D(A,B) = n \cdot h(a,b) \neq n \cdot h(b,a) = D(B,A)$. Absurd.

3. $h(0,\cdot)$ is monotonous no decreasing in $[0,1]$. Let be $0 \le a \le b \le 1$, then $\emptyset \subseteq A \subseteq B$, and $n \cdot h(0,a) = D(\emptyset, A) \le D(\emptyset, B) = n \cdot h(0,b)$.

4. $h(a{+}t, b{+}t) = h(a,b)$, with $t \in [-1,1]$, $a{+}t, b{+}t \in [0,1]$.

   (a) if $t \le 0$ and $a{+}t, b{+}t \in [0,1]$, let be $c = 1{+}t$. $D(A \cap C, B \cap C) = n \cdot h(a+c-1, b+c-1) \le D(A,B) = n \cdot h(a,b)$. Then $h(a+t, b+t) \le h(a,b)$.

   (b) if $t > 0$ and $a+t, b+t \in [0,1]$. Let be $c = t$, $D(A \cup C, B \cup C) = n \cdot h(a+c, b+c) \le D(A,B) = n \cdot h(a,b)$. Then $h(a+t, b+t) \le h(a,b)$.

   Since a, b and t are arbitrary, then it follows that $h(a+t, b+t) = h(a,b)$

5. $h(a,b) = h(|a-b|, 0)$. If $a \le b$, then $h(a,b) = h(a{-}a, b{-}a) = h(0, b{-}a)$. If $b \le a$, then $h(a,b) = h(a-b, b-b) = h(a-b, 0) = h(0, a-b)$.

6. Let be $g(a) = h(a,0)$. Then $g(0) = 0$ and $g(\cdot)$ is monotonous no decreasing in $[0,1]$.

Thus

$$D(A,B) = \sum_{i=1}^{n} h(A(x_i), B(x_i)) =$$
$$= \sum_{i=1}^{n} h(|A(x_i) - B(x_i)|, 0) =$$
$$= \sum_{i=1}^{n} g(|A(x_i) - B(x_i)|)$$

To prove the counterpart, let be $D(A,B) = \sum_{i=1}^{n} g(|A(x_i) - B(x_i)|)$, with $g(0) = 0$ and g monotonous no decreasing. Trivially, $D(A,A) = 0$ and $D(A,B) = D(B,A)$. Let be $\Omega_I = \{x \in \Omega | A(x) + C(x) \ge 1,\ B(x) + C(x) \ge 1\}$, $\Omega_{II} = \{x \in \Omega | A(x) + C(x) \ge 1,\ B(x) + C(x) < 1\}$, $\Omega_{III} = \{x \in \Omega | A(x) + C(x) < 1,\ B(x) + C(x) \ge 1\}$, $\Omega_{IV} = \{x \in \Omega | A(x) + C(x) < 1,\ B(x) + C(x) < 1\}$. Thus

$$D(A \cup C, B \cup C) =$$
$$= \sum_{x \in \Omega_I} g(|1-1|) +$$
$$+ \sum_{x \in \Omega_{II}} g(|1 - (B(x) + C(x)|) +$$
$$+ \sum_{x \in \Omega_{III}} g(|(A(x_i) + C(x)) - 1|) +$$
$$+ \sum_{x \in \Omega_{IV}} g(|(A(x) + C(x)) - (B(x) + C(x))|)$$

Since g is monotonous, then if $a + c \ge 1 \ge b + c$, it follows $g(|1 - (b+c)|) \le g(|(a+c) - (b+c)|)$, and so on. Therefore

$$D(A \cup C, B \cup C) \le D(A,B).$$

Similar proof for $D(A \cap C, B \cap C) \le D(A,B)$. ■

We extend the results of [10].

Theorem 2 *Let μ be a fuzzy measure. Then there exits a divergence measure D such that*

$$\mu(A) = D(A, \emptyset)$$

Proof. Let be $(|A - B|)(x) = |A(x) - B(x)|$, $\forall x \in \Omega$, and $D(A,B) = \mu(|A - B|)$. D must verify the axioms **d1-d3** to be a divergence measure. Obviously, $D(A,B) = D(B,A)$ and $D(A,A) = 0$. To prove that $D(A \cup C, B \cup C) \le D(A,B)$, let us see that $(|A \cup C - B \cup C|)(x) \le (|A - B|)(x)$, $\forall x \in \Omega$. Since $(|A \cup C - B \cup C|)(x) = |\min\{A(x) + C(x), 1\} - \min\{B(x) + C(x), 1\}|$, by same reasons that in the counterpart of Theorem 1, it holds that $(|A \cup C - B \cup C|)(x) \le (|A - B|)(x)$, $\forall x \in \Omega$. Analogous proof for $D(A \cap C, B \cap C) \le D(A,B)$. Therefore the axiom **d3** holds and D is a divergence measure such that $\mu(A) = D(A, \emptyset)$. ■

Theorem 3 *Let D be a local divergence measure and g its generator function. Then $g(a_1 + a_2) \le g(a_1) + g(a_2)$, $\forall a_1, a_2 \in [0,1]$, if and only if D is a pseudo-metric.*

Proof. Let us prove that

$$D(A,B) \le D(A,C) + D(B,C).$$

Since D is local, then it must verify that

$$\sum_{i=1}^{n} g(|A(x_i) - B(x_i)|) \le$$
$$\le \textstyle\sum_{i=1}^{n} g(|A(x_i) - C(x_i)|) + g(|C(x_i) - B(x_i)|)$$

If $C(x) \le \min\{A(x), B(x)\}$ or $\max\{A(x), B(x)\} \le C(x)$ it holds trivially by the monotonicity of g. If $A(x) \le C(x) \le B(x)$, let be $a_1 = |A(x) - C(x)|$, $a_2 = |B(x) - C(x|$. Then

$$g(|A(x) - B(x)|) \le$$
$$\le g(|A(x) - C(x)|) + g(|C(x) - B(x)|)$$

Analogous proof for $B(x) \le C(x) \le A(x)$.

The counter part is obvious ($A(x) = a_1 + a_2$, $B(x) = 0$, $C(x) = a_2$). ■

3 CONCLUSIONS

In this paper we have introduced an axiomatic approach to the cluster validity problem using basic concepts (axioms **d1**-**d3**) and the connectives T_∞ and S_∞.

We gave a characterization of divergence measures when all data are equally important, also we have remarked its relation with the pseudodistance and we have obtained another point of view of fuzzy measures based on divergence measures. Our interpretation says that a fuzzy measure $\mu(A)$, the degree of evidence or belief that a particular element belongs in the set A, is the divergence measure between A and $\emptyset$, the degree of evidence or belief that a particular element belongs in the set A and not in $\emptyset$.

We think that a divergence measure is suitable to quantify the difference between two fuzzy sets in the context of fuzzy partitions.

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