On measuring μ-T-conditionality of fuzzy relations (Part II)

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Abstract

This paper continues and finishes other works toward proposing a method to measure the μ-T-conditionality character of any fuzzy relation for any continuous t-norm T.

Keywords: fuzzy relation, conditionality, conditionalized fuzzy relation, conditionality measure of a fuzzy relation.

1 INTRODUCTION

The μ-T-conditionality property of fuzzy relation seems to be very interesting in order to make fuzzy inference that generalises the Modus Ponens property.

Let E_1, E_2 be two sets, let E be the set $E_1 \cup E_2$, and let $\mu: E \rightarrow [0,1]$ be a fuzzy set and let T be a continuous t-norm. A fuzzy relation R: $E_1 \times E_2 \rightarrow [0,1]$ holds the μ-T-conditionality property if and only if $T(\mu(a), R(a,b)) \leq \mu(b)$ for all (a, b) in $E_1 \times E_2$.

There were proposed [1] two ways for measuring μ-T-conditionality of fuzzy relations. A first way calculates a generalised distance between a fuzzy relation R and the greatest μ-T-conditional relation that is contained in R. The other way measures the difference between $T(\mu(a), R(a,b))$ and $\mu(b)$ in all (a,b) in which R is not μ-T-conditional.

This paper completes a partial result obtained in [2] which proves that when T is the t-norm minimum or any archimedean t-norm, and when a generalised distance defined from a residuated operator of the T-norm is used, both methods proposed in [1] give the same measure values of μ-T-conditionality of fuzzy relations. This paper proves such result also for ordinal sums, so the result is finally held for all continuous t-norms.

2 PRELIMINARIES

1. Let T_R^{μ} be the fuzzy relation defined by $T_R^{\mu}(a,b)=T(\mu(a), R(a,b))$.
2. From the well known operation in [0,1], $J^T(x, y)=\mathrm{Sup}\ \{z \ / \ T(x, z) \leq y\}$, lets define J_{μ}^T as the fuzzy relation $J_{\mu}^T(a, b)=J^T(\mu(a), \mu(b))$.
3. The **μ-T-conditionalized relation** of R is defined as:

$R_C^{\mu\text{-}T}(a,b) = \mathrm{Min}(R(a,b), J_{\mu}^T(a,b))$

$$= \begin{cases} R(a, b) & \text{if } T_R^{\mu}(a, b) \leq \mu(b) \\ J_{\mu}^T(a, b) & \text{otherwise} \end{cases}$$

4. A Generalised Metric Space is a triplet (E, ℘, m) where E is a set, ℘=(A, S, ≤, e) is a commutative semigroup with neutral element e, and m is a **S-distance**, which is an application m: E×E→A such that:
   1) m(a, a) = e, for all a in E
   2) $m(a, c) \leq S(m(a, b), m(b, c))$, for all a,b,c in E

(S-triangular inequality)

Let T* be the dual t-conorm of a t-norm T, defined by $T^*(x,y)=1-T(1-x, 1-y)$.

Let ℑ*=([0, 1], T*, ≤, 0) be a commutative ordered semigroup with neutral element 0.

Given a relational structure (E, J), if J is a T-preorder then the function $d(a, b)=1- J^T(a, b)$ is a T*-distance in the generalised metric space ([0,1], ℑ*, d).

5. It is defined the μ-T-unconditionality region of a fuzzy relation, **$INC_T^{\mu}(R)$**, as the subset of $E_1 \times E_2$ in which R is not μ-T-conditional, that is:

$INC_T^{\mu}(R)$ $=\{(a, b) \in E_1 \times E_2$ such that $T(\mu(a), R(a, b)) > \mu(b)\}$

$= \{(a, b)$ such that $T_R^{\mu}(a, b) > \mu(b)\}$

$= \{(a, b) \in E_1 \times E_2$ such that $R(a, b) > J_{\mu}^T(a, b)\}$

6. The family of t-norms of a triangular t-norm T is the set of t-norms defined as

$T_{\varphi}(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y))$, given any continuous, strictly increasing function φ: [0, 1] x [0, 1] → [0, 1] such that φ(0)=0 and φ(1)=1.

Let $\varphi\mu$: E → [0, 1] be the fuzzy set defined by $\varphi\mu(a) = \varphi(\mu(a))$. Let φR: $E_1 \times E_2 \rightarrow [0, 1]$ be the fuzzy relation defined by $\varphi R(a, b) = \varphi(R(a, b))$.

7. A t-norm T is an ordinal sum if there exist a finite or numerable collection of archimedean t-norms $\{T_i \,/\, i\in J\}$ and a collection of disjoint intervals $\{(a_i,b_i)\}/\, i\in J\}$ in $[0,1]$ such that T(x,y)=

$$\begin{cases} a_i + (b_i - a_i) T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y)\in[a_i, b_i]^2 \\ \operatorname{Min}(x,y) & \text{otherwise} \end{cases}$$

The proofs of both following theorems can be found in [2].

Theorem 1.
Given a fuzzy relation R, a fuzzy set μ, a function φ and a t-norm T, it holds that

$(a, b)\in INC_{T\varphi}^{\mu}(R)$ if and only if $(a, b)\in INC_T^{\varphi\mu}(\varphi R)$.

Given a fuzzy set $\mu: E\to[0,1]$, it is defined the fuzzy relation $\mu_2:E_1\times E_2\to[0,1]$ as $\mu_2(a, b)= \mu(b)$.

Theorem 2
Given the t-norm minimum or any archimedean t-norm T, for all $(a, b) \in E_1\times E_2$, the distance $1\text{-}J^T$ between a fuzzy relation R in the point (a, b) and its μ-T-conditionalized relation in (a, b) is the same that the distance $1\text{-}J^T$ between T_R^{μ} (a, b) and $\mu_2(a, b)$. This is, it holds that

$J^T(R(a, b), J_{\mu}^T(a, b))=J^T(T_R^{\mu}(a,b),\mu_2(a,b))$.

3 THE CASE OF ORDINAL SUMS

Lemma 1
Let T be an ordinal sum. If $x, y \in [a_i, b_i]$, then $T(x, y) \in [a_i, b_i]$.
Proof
T is monotonous, $T(a_i, a_i) = a_i$, $T(b_i, b_i) = b_i$, so $a_i \le T(x, y) \le b_i$. z

Lemma 2
Let T be an ordinal sum defined through a collection of archimedean t-norms $\{T_i \,/\, i\in J\}$ and a collection of disjoint intervals $\{(a_i, b_i)\}/\, i\in J\}$. The residuated operation of T is $J^T(x, y) = \sup\{z \,/\, T(x, z) \le y) =$

$$\begin{cases} 1 & \text{if } x\le y \\ y & \text{if } x>y \text{ and } (x,y)\notin[a_i,b_i] \quad \text{for all } i\in J \\ a_i+(b_i-a_i)J^{T_i}\left(\frac{x-a_i}{b_i-a_i},\frac{y-a_i}{b_i-a_i}\right) & \text{if } x>y \text{ and } (x,y)\notin[a_i,b_i] \end{cases}$$

Proof
1) **If $x \le y$,** any residuated operation of a t-norm takes the value 1.

2) **If x>y** and **$(x, y) \notin [a_i, b_i]^2$,** then $J^T(x,y) = y$. This is because

2.1) if $x \notin [a_i, b_i]$, then $J^T(x, y) = \sup\{z/ T(x, z)\le y\} = \sup\{z/ \operatorname{Min}(x, z)\le y\} = y$.

2.2) if $x\in[a_i, b_i]$ and $y\notin[a_i, b_i]$, then $z = J^T(x, y)$ is not in $[a_i, b_i]$, because y<x, so $y<a_i$ and if $z = J^T(x, y)$ would be in $[a_i, b_i]$, then, by lemma 1, T(x, z) would also be in $[a_i, b_i]$ which contradicts $T(x, z)\le y$. As $z\notin[a_i, b_i]$ the t-norm must be the minimum and $J^T(x, y) = \sup\{z \,/\, T(x, z) \le y\} = \sup\{z \,/\, \operatorname{Min}(x, z)\le y\} = y$.

3) **If x>y** and **$(x, y)\in[a_i, b_i]^2$,** then $J^T(x, y)\in[a_i, b_i]$. This holds because if $z =J^T(x, y)$ would not be in $[a_i, b_i]$, then:
$J^T(x, y) = \sup\{z \,/\, T(x, z) \le y\} = \sup\{z \,/\, \operatorname{Min}(x, z) \le y\} = y$, which contradicts with $y\in[a_i, b_i]$. So, in this case:

$J^T(x, y) = \sup\{z/T(x,z)\le y\}$

$= \sup\{z/ a_i+(b_i - a_i)T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{z-a_i}{b_i-a_i}\right)\le y \}$

$= \sup\{z/ T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{z-a_i}{b_i-a_i}\right)\le \frac{y-a_i}{b_i-a_i} \}$

$= \{z \,/\, \frac{z-a_i}{b_i-a_i} =J^{Ti}\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) \}$

$= a_i+(b_i-a_i)\, J^{Ti}\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)$. z

Definition 1.
Let R be a fuzzy relation. The fuzzy relation $\mathbf{R_{I_i}}$ restricted from R to the interval $I_i=[a_i, b_i]$, is defined by

$R_{I_i}(a, b)= \frac{R(a, b)-a_i}{b_i-a_i}$.

Let μ be a fuzzy set. The fuzzy set μ_{Ii} restricted from μ to the interval $[a_i, b_i]$, is defined by

$$\mu_{Ii}(a)= \frac{\mu(a) - a_i}{b_i - a_i}.$$

Theorem 3.

Let R be a fuzzy relation, let μ be a fuzzy set, and let T be an ordinal sum, then

$J^T(R(a, b), J_\mu{}^T(a, b)) = J^T(T_R{}^\mu(a, b), \mu_2(a,b))$ for all (a,b) in $E_1{\times}E_2$.

Proof

1) If $\mathbf{R(a, b) \leq J_\mu{}^T(a, b)}$, it is, if $(T)_R{}^\mu(a, b) \leq \mu_2(a, b)$, then $J^T(R(a, b), J_\mu{}^T(a, b)) = J^T(T_R{}^\mu(a, b), \mu_2(a, b)) = 1$.

It includes the case $\mathbf{\mu(a) \leq \mu(b)}$.

On the other cases, it is supposed that

$\mathbf{R(a, b) > J_\mu{}^T(a, b)}$, it is, $\mathbf{T_R{}^\mu(a, b) > \mu_2(a, b)}$ and $\mathbf{\mu(a) > \mu(b)}$.

2) If $\mathbf{\mu(b) \notin [a_i, b_i]}$, then, by lemma 2, case 2.2 it holds that:

$J^T(R(a, b), J_\mu{}^T(a, b))$

$= J^T(R(a, b), J^T(\mu(a), \mu(b))$

$= J^T(R(a, b), \mu(b))$

$= \mu(b)$

$=J^T(T_R{}^\mu(a, b), \mu(b))$

$=J^T((T)_R{}^\mu(a, b), \mu_2(a, b))$.

3) If $\mathbf{\mu(a), \mu(b) \in [a_i, b_i]}$ and $\mathbf{R(a,b) \notin [a_i, b_i]}$, then, by lemma 2 it holds that

$J^T(R(a, b), J_\mu{}^T(a, b))$

$= J_\mu{}^T(a, b)$

$= J^T(\mu(a), \mu(b))$ (*)

$= J^T(Min(\mu(a), R(a, b)), \mu(b))$

$= J^T(T(\mu(a), R(a, b)), \mu(b))$

$= J^T(T_R{}^\mu(a, b), \mu_2(a, b))$.

(*) $T(\mu(a), R(a, b))>\mu(b)$, so $R(a, b))>\mu(b)$, but $R(a, b) \notin [a_i, b_i]$, so $a_i \leq \mu(b) < \mu(a) \leq b_i < R(a, b))$, and $Min(\mu(a), R(a, b))=\mu(a)$.

4) If $\mathbf{R(a,b), \mu(b)}$ are in $\mathbf{[a_i, b_i]}$ and $\mathbf{\mu(a) \notin [a_i, b_i]}$, then

$J^T(R(a, b), J_\mu{}^T(a, b))$

$= J^T(R(a, b), J^T(\mu(a), \mu(b))$

$= J^T(R(a, b), \mu(b))$ (*)

$= J^T(Min(\mu(a), R(a, b)), \mu(b))$

$= J^T(T(\mu(a), R(a, b)), \mu(b))$

$= J^T(T_R{}^\mu(a, b), \mu_2(a, b))$

(*) $T(\mu(a), R(a, b))>\mu(b)$, so $R(a, b))>\mu(b)$, and $a_i \leq \mu(b) < R(a,b) \leq b_i < \mu(a)$, so $Min(\mu(a), R(a, b)) = R(a, b)$.

5) If $\mathbf{\mu(b) \in [a_i, b_i]}$ and $\mathbf{\mu(a), R(a, b)}$ are not in $\mathbf{[a_i, b_i]}$, then

$J^T(R(a, b), J_\mu{}^T(a, b))$

$= J^T(R(a, b), J^T(\mu(a), \mu(b))$

$= \mu(b)$ (*)

$= J^T(T(\mu(a), R(a, b)), \mu(b))$

$= J^T(T_R{}^\mu(a, b), \mu_2(a, b))$

(*) By lemma 1, as $\mu(a), R(a, b)$ are not in $[a_i, b_i]$, then $T(\mu(a), R(a, b))$ is not in $[a_i, b_i]$.

6) If $\mathbf{\mu(a), \mu(b), R(a, b) \in [a_i, b_i]}$, then by lemma 2, case 3, it holds that:

$J^T(R(a, b), J_\mu{}^T(a, b))$

$$= J^T\left(R(a,b), a_i+(b_i-a_i) J^{Ti}\left(\frac{\mu(a)-a_i}{b_i-a_i}, \frac{\mu(b)-a_i}{b_i-a_i}\right)\right)$$

$$= a_i+(b_i-a_i)J^{Ti}\left(\frac{R(a,b)-a_i}{b_i-a_i}, J^{Ti}\left(\frac{\mu(a)-a_i}{b_i-a_i}, \frac{\mu(b)-a_i}{b_i-a_i}\right)\right)$$

$$= a_i+(b_i-a_i) J^{Ti}\left(R_{Ii}(a,b), J^{Ti}_{\mu_{Ii}}(a,b)\right) \quad (*)$$

$$= a_i+(b_i-a_i) J^{Ti}\left((T_i)_{R_{Ii}}^{\mu_{Ii}}(a,b), \mu_{Ii}(b)\right)$$

$$= a_i+(b_i-a_i)J^{Ti}\left(T_i\left(\frac{\mu(a)-a_i}{b_i-a_i}, \frac{R(a,b)-a_i}{b_i-a_i}\right), \frac{\mu(b)-a_i}{b_i-a_i}\right)$$

$$= J^T\left(a_i+(b_i-a_i)T_i\left(\frac{\mu(a)-a_i}{b_i-a_i}, \frac{R(a,b)-a_i}{b_i-a_i}\right), \mu(b)\right) \quad (**)$$

$= J^T(T(\mu(a), R(a, b)), \mu(b))$

$= J^T(T_R{}^\mu(a, b), \mu_2(a, b))$.

(*) For archimedean t-norms it holds that $J^T(R(a, b), J_\mu{}^T(a, b))=J^T(T_R{}^\mu(a,b), \mu_2(a,b))$, so, for any t-norm T_i, the fuzzy relation R_{Ii} restricted to interval $[a_i, b_i]$ and the fuzzy set μ_{Ii} restricted to $[a_i, b_i]$ holds that $J^{Ti}\left(R_{Ii}(a,b), J^{Ti}_{\mu_{Ii}}(a,b)\right) = J^{Ti}((T_i)^{\mu_{Ii}}_{R_{Ii}}(a, b), \mu_{Ii}(b))$.

(**) By lemma 1, $T(\mu(a), R(a, b)) =$

$a_i+(b_i-a_i)T_i\left(\frac{\mu(a)-a_i}{b_i-a_i}, \frac{R(a,b)-a_i}{b_i-a_i}\right) \in [a_i, b_i]$. z

Theorem 4.

Let R be a fuzzy relation, let μ be a fuzzy set, and let T be any continuous t-norm, then $J^T(R(a, b), J_\mu{}^T(a, b)) = J^T(T_R{}^\mu(a, b), \mu_2(a,b))$ for all (a, b) in $E_1 \times E_2$

Proof

It follows from theorems 2 and 3.

Theorem 5

Given any continuous t-norm T, for all (a, b) in $E_1 \times E_2$, the distance $1-J^T$ between a fuzzy relation R in the point (a, b) and its μ-T-conditionalized relation in (a, b) is the same that the distance $1-J^T$ between $T_R{}^\mu$ (a, b) and $\mu_2(a, b)$.

Proof

It follows from theorem 4

4 EXAMPLES

Similarly to the examples in [2], it is given the following examples:

1. When using the fuzzy T*-distance between two fuzzy relation R and R' defined by $d_T(R, R') = \sup_{(a,b)\in E_1\times E_2} \{1-J^T(R(a, b), R'(a, b))\}$, for any continuous t-norm, the distance d_T between a fuzzy relation R and its μ-T-conditionalized fuzzy relation is equal to the distance d_T between $T(\mu(a), R(a,b))$ and $\mu_2(a, b)=\mu(b)$.

That is, for any continuous t-norm, a μ-T-unconditionality measure of fuzzy relation might be calculated as

$\sup_{(a,b)\in E_1\times E_2} \{1-J^T(R(a, b), J_\mu{}^T(a, b))\}$ or as

$\sup_{(a,b)\in E_1\times E_2} \{1- J^T((T)_R{}^\mu(a, b), \mu_2(a,b))\}$.

2. For any continuos t-norm T, the order k moment of μ-T-unconditionality of a fuzzy relation might be calculated as

$\iint_{(a,b)\in E_1\times E_2} (1-J^T(R(a, b), J_\mu{}^T(a, b)))^k$ or as

$\iint_{(a,b)\in E_1\times E_2} (1- J^T((T)_R{}^\mu(a, b), \mu_2(a,b)))^k$.

5 CONCLUSION

It is proved in [2] that given an archimedean t-norm T or the t-norm minimum, both ways described in [1] to measure the μ-T-conditionality property of fuzzy relations, give the same results when using a distance between fuzzy relations defined from the distance $1-J^T$. This paper extends the result for ordinal sums, and so, the result is proven for all continuous t-norms.

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