Axiomatizing t-Norm Based Logic

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Abstract

In fuzzy logic in wider sense, t-norms got a prominent rôle in recent times.

In many-valued logic, the Łukasiewicz systems, the Gödel sytems, and also the product logic are t-norm based. The present paper discusses the more general problem of the adequate axiomatizability for such t-norm based logical systems in general, surveying results of the last years.

Keywords: fuzzy logic, many-valued logic, t-norms, t-norm based connectives.

1 Semantic approach

There is an important difference to the standard approach toward semantically based systems of many-valued logic: in the area of t-norm based systems there is no single, "standard" semantical matrix for the general approach.

The most appropriate way out thus shall be: to find some suitable class(es) of algebraic structures which can be used to characterize these logical system(s).

An important restriction has to be met: to left continuous (or even to continuous) t-norms, because one (usually) likes to have an R-implication connective together with a basic t-norm.

As it seems, the notions of left continuity and of continuity need the reference to the notion of limit. And this is not (genuinely) an algebraic notion.

Hence it would be nice to find – algebraically characterizable – classes of algebraic structures which "approximate" well the class of algebraic structures determined by the continuous, or by the left continuous t-norms.

The following is structurally important for t-norms:

- $\langle [0,1], \boldsymbol{t}, 1\rangle$ is a commutative semigroup with a neutral element, i.e. a commutative monoid,
- $\leq$ is a (lattice) ordering in $[0,1]$ which has a universal lower bound and a universal upper bound,
- both structures "fit together": $\boldsymbol{t}$ is non-decreasing w.r.t. this lattice ordering.

Thus one has to consider (complete) abelian lattice-ordered monoids as the truth degree structures for the t-norm based systems.

In general, abelian lattice-ordered monoids may have different elements as the universal upper bound of the lattice and the neutral element of the monoid. This is not so for the t-norm based systems, they have an *integral* abelian lattice-ordered monoid as truth degree structure.

One also likes to have the t-norm combined with a pseudo-complementation, its R-implication operator: i.e. the abelian lattice-ordered monoid formed by the truth degree structure has also to be a *residuated* one.

At all, hence, we consider *residuated lattices*, i.e. algebraic structures $\langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$ such that L is a lattice under $\cap, \cup$ with zero 0 and unit 1, and an abelian lattice-ordered monoid under $*$ with neutral element 1, and such that the operations $*$ and $\rightarrowtail$ form an adjoint pair, i.e. satisfy

$$z \leqslant (x \rightarrowtail y) \quad \Leftrightarrow \quad x * z \leqslant y.$$

It is nice to recognize that the adjointness condition is in the present setting the suitable algebraic equivalent of the analytical notion of left continuity.

Proposition 1 *For any t-norm* $\boldsymbol{t}$ *one has that* $\boldsymbol{t}$ *and its R-implication form an adjoint pair iff* $\boldsymbol{t}$ *is left continuous (in both arguments).*

And also the continuity of the basic t-norm has an algebraic equivalent: the property of divisibility for the commutative lattice-ordered monoids.

Definition 1 $\langle L, *, 1, \leqslant \rangle$ *is* divisible *iff for all* $a, b \in L$ *with* $a \leqslant b$ *there exists some* $c \in L$ *with* $a = b * c$.

For residuated lattices one has another characterization of divisibility:
$\langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$ is divisible iff one has $a \cap b = a * (a \rightarrowtail b)$ for all $a, b \in L$.

Proposition 2 *A t-norm based residuated lattice* $\langle [0,1], \min, \max, \boldsymbol{t}, \mathsf{seq}_{\boldsymbol{t}}, 0, 1\rangle$ *is divisible iff* $\boldsymbol{t}$ *is continuous.*

A further restriction is suitable because each t-norm based residuated lattice is linearly ordered, and thus makes the wff $(\varphi \rightarrow_{\boldsymbol{t}} \psi) \vee (\psi \rightarrow_{\boldsymbol{t}} \varphi)$ valid.

So finally we call *BL-algebras* divisible residuated lattices which also satisfy the *pre-linearity* condition:
$(x \rightarrowtail y) \cup (y \rightarrowtail x) = 1$.

2 Syntactic approach

Now we consider the problems of syntactic characterizations of the classes of all wffs which are valid in all residuated lattices, or in all BL-algebras. For more details the reader may e.g. consult [2].

2.1 Monoidal logic

The axiom system $\mathsf{Ax}_{\mathsf{ML}}$ of Höhle [6, 7] for the class of wffs which are valid in all residuated lattices has the axiom schemata:

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)),$$
$$\varphi \rightarrow \varphi \vee \psi,$$
$$\psi \rightarrow \varphi \vee \psi,$$
$$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)),$$
$$\varphi \wedge \psi \rightarrow \varphi,$$
$$\varphi \,\&\, \psi \rightarrow \varphi,$$
$$\varphi \wedge \psi \rightarrow \psi,$$
$$\varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi,$$
$$(\varphi \,\&\, \psi) \,\&\, \chi \rightarrow \varphi \,\&\, (\chi \,\&\, \chi),$$
$$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)),$$
$$(\varphi \rightarrow (\chi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \chi \rightarrow \chi),$$
$$(\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)),$$
$$\varphi \,\&\, \neg\varphi \rightarrow \psi,$$
$$(\varphi \rightarrow \varphi \,\&\, \neg\varphi) \rightarrow \neg\varphi,$$

and has as its (only) inference rule the rule of detachment (w.r.t. the implication connective $\rightarrow$).

The logical calculus which is constituted by this axiom system and its inference rule, and which has the standard notion of derivation, shall be denoted by $\mathbb{K}_{\mathsf{ML}}$.

Proposition 3 *The calculus* $\mathbb{K}_{\mathsf{ML}}$ *is sound, i.e. proves only such formulas which are valid in all residuated lattices.*

Corollary 4 *The* Lindenbaum *algebra of the calculus* $\mathbb{K}_{\mathsf{ML}}$ *is a residuated lattice.*

Theorem 5 (Completeness) *For a wff* H *of* $\mathcal{L}_{\boldsymbol{t}}$ *there are equivalent:*
(i) *H is derivable within the logical calculus* $\mathbb{K}_{\mathsf{ML}}$*;*
(ii) *H is valid in all residuated lattices.*

Proof: The part $(i) \Rightarrow (ii)$ here is just the soundness of the calculus $\mathbb{K}_{\mathsf{ML}}$. The part $(ii) \Rightarrow (i)$ on the other hand results from the fact that the Lindenbaum algebra of $\mathbb{K}_{\mathsf{ML}}$ is a residuated lattice and has the class of all $\mathbb{K}_{\mathsf{ML}}$-derivable wffs as its unit element, i.e. as the universal upper bound of its lattice part and at the same time as the neutral element of its monoidal part. Therefore each non-$\mathbb{K}_{\mathsf{ML}}$-derivable wff is not valid in at least one residuated lattice. □

This monoidal logic can be specialized toward quite different systems of (non-classical) logic by adding suitable further axioms.
Remark: The condition of pre-linearity is satisfied in all MV-algebras.

Theorem 6 *From the monoidal logic one gets adequate axiomatizations of:*
(a) the ŁUKASIEWICZ *system* L_∞ *if one adds the axiom schemata*

$$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)\,,$$
$$\neg\neg\varphi \to \varphi\,;$$

(b) the GÖDEL *system* G_∞ *if one adds the axiom schemata*

$$\varphi \to \varphi \,\&\, \varphi\,,$$
$$(\varphi \to \psi) \vee (\psi \to \varphi)\,;$$

(c) the product logic Π *if one adds the axiom schemata*

$$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \chi)\,,$$
$$(\varphi \to \psi) \vee (\psi \to \varphi)\,,$$
$$\varphi \wedge \psi \to \bar{0}\,,$$
$$\neg\neg\chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))\,.$$

This monoidal logic covers also further systems of non-classical logic. The most important is GIRARD's *commutative linear logic* which may be characterized, cf. [7], by the class of all those residuated lattices which satisfy the law: $a = (a \rightarrowtail 0) \rightarrowtail 0$.

Proposition 7 *An adequate axiomatization of the commutative linear logic of* GIRARD *is given by the axiom system* $\mathsf{Ax_{ML}}$ *together with the additional schema:* $\neg\neg\varphi \to \varphi$.

And also classical (propositional) logic can be axiomatized by an extension of $\mathsf{Ax_{ML}}$.

Proposition 8 *An adequate axiomatization of classical propositional logic is given by the axiom system* $\mathsf{Ax_{ML}}$ *together with the additional schemata*

$$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)\,,$$
$$\neg\neg\varphi \to \varphi\,,$$
$$\varphi \to \varphi \,\&\, \varphi\,.$$

2.2 Basic logic

The axiomatization of HÁJEK [3] for the basic logic, i.e. for the class of all wffs which are valid in all BL-algebras, is given in a language which has as basic vocabulary the connectives $\to, \&$ and the truth degree constant $\bar{0}$, taken in each BL-algebra $\langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$ as the operations $\rightarrowtail, *$ and the element 0. Then basic logic has as axiom system $\mathsf{Ax_{BL}}$ the following schemata:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\,,$$
$$\varphi \,\&\, \psi \to \varphi\,,$$
$$\varphi \,\&\, \psi \to \psi \,\&\, \varphi\,,$$
$$\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\chi \to \varphi)\,,$$
$$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)\,,$$
$$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))\,,$$
$$((\varphi \to \psi) \to \chi) \to (((\chi \to \varphi) \to \chi) \to \chi)\,,$$
$$\bar{0} \to \varphi\,,$$

and has as its (only) inference rule the rule of detachment (w.r.t. the implication connective $\to$).

The logical calculus which is constituted by this axiom system and its inference rule, and which has the standard notion of derivation, shall be denoted by $\mathbb{K}_{\mathsf{BL}}$.

Usually the language is extended by definitions of additional connectives and a further truth degree constant:

$$\varphi \wedge \psi \;=_{\text{def}}\; \varphi \,\&\, (\varphi \to \psi)\,,$$
$$\varphi \vee \psi \;=_{\text{def}}\; ((\varphi \to \chi) \to \psi) \wedge ((\chi \to \varphi) \to \varphi)\,,$$

together with $\bar{1} =_{\text{def}} \bar{0} \to \bar{0}$.

Calculations (in BL-algebras) show that the additional connectives $\wedge, \vee$ just have the BL-algebraic operations $\cap, \cup$ as their truth degree functions.

It is a routine matter, but a bit tedious, to check that this logical calculus $\mathbb{K}_{\mathsf{BL}}$ is sound, i.e. derives only such formulas which are valid in all BL-algebras.

Corollary 9 *The axioms of the monoidal logic are* $\mathbb{K}_{\mathsf{BL}}$*-derivable.*

Corollary 10 *The* LINDENBAUM *algebra of the logical calculus* $\mathbb{K}_{\mathsf{BL}}$ *is a BL-algebra.*

Theorem 11 (Completeness) *For a wff* H *of* $\mathcal{L}_t$ *there are equivalent:*
(*i*) *H is derivable within the logical calculus* $\mathbb{K}_{\mathsf{BL}}$*;*
(*ii*) *H is valid in all BL-algebras;*
(*iii*) *H is valid in all linearly ordered BL-algebras, i.e. in all BL-chains.*

Theorem 12 *From the basic logic one gets adequate axiomatizations of:*
(*a*) *the system* L_∞ *if one adds the axiom schema:*

$$\neg\neg\varphi \to \varphi\,;$$

(*b*) *the system* G_∞ *if one adds the axiom schema:*

$$\varphi \to \varphi \,\&\, \varphi\,;$$

(*c*) *the product logic* Π *if one adds the axiom schemata*

$$\begin{aligned} &\varphi \wedge \psi \to \bar{0}\,,\\ &\neg\neg\chi \to\\ &\quad ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))\,. \end{aligned}$$

2.3 The logic of continuous t-norms

From the preceding considerations it is clear that all the logically valid wffs of monoidal (resp. basic) logic are valid in all t-norm based residuated lattices with a left continuous (resp. a continuous) t-norm.

Problem thus are whether there exist wffs (i) which are valid in all t-norm based residuated lattices with a left continuous t-norm but are not theorems of monoidal logic, and (ii) which are valid in all t-norm based residuated lattices with a continuous t-norm but are not theorems of basic logic.

The first problem is actually an open one. But the second problem has a negative answer, given by the following theorem.

Theorem 13 (Completeness) *The class of all wff which are provable in basic logic coincides with the class of all wffs which are logically valid in all t-norm based residuated lattices with a continuous t-norm.*

The main steps in the proof are to show (i) that each BL-algebra is a subdirect product of BL-chains, i.e. of linearly ordered BL-algebras, and (ii) that each BL-chain is the ordinal sum of BL-chains which are either trivial one-element BL-chains, or linearly ordered MV-algebras, or linearly ordered product algebras (as introduced in [5]), such that (iii) each such ordinal summand is locally embeddable into a t-norm based residuated lattices with a continuous t-norm, cf. [1, 4].

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