AGGREGATION FUNCTIONS AND FUZZY MEASURES: THE MULTI-DIMENSIONAL CASE

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Summary

This paper is devoted to the study of multi-dimensional fuzzy measures and integrals from the point of view of aggregation operators. A necessary and sufficient condition for a multi-dimensional discrete Choquet (and Sugeno) integral to be a multi-dimensional aggregation function (MAF) is given. The particular case of fuzzy measures generated by a t-conorm is specially studied.

Keywords: Multi-dimensional aggregation function, fuzzy measure, discrete Sugeno and Choquet integrals, t-conorm, iteration.

1 INTRODUCTION

Fuzzy measures were introduced by M. Sugeno in 1974 in order to express a grade of fuzziness in the same way that probability measures express a grade of randomness. The Sugeno fuzzy integrals are the functionals with monotonicity defined by using fuzzy measures. Later on, Murofushi and Sugeno proposed another type of fuzzy integral, the Choquet integral, based on the Capacity Theory developed by G. Choquet in 1953. Many authors have investigated on the characterization of Sugeno and Choquet integrals (P. Wakker, D. Schmeidler, D. Dubois, H. Prade, R. Sabbadin, J. L. Marichal). Both fuzzy integrals have proved to be very useful as aggregation operators.

In this paper, we deal with the multi-dimensional aspect of fuzzy measures and fuzzy integrals, and we give conditions for such a multi-dimensional functions to be a Multi-dimensional Aggregation Function. This kind of aggregation functions have been already studied by G. Mayor and T. Calvo in [5].

2 PRELIMINARIES

In this section we present the definitions and basic results we will use throughout this paper.

Let us denote by E the set $E = \bigcup_{n\geq 1} [0,1]^n$ of all the ordered lists formed by elements of $[0,1]$.

Definition 1 *Given* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ *elements of* E*, we define the following relations:*

- $x \leq_\pi y$ *if and only if* $n = m$ *and* $x_1 \leq y_1, \ldots, x_n \leq y_n$ *(product order)*
- $x \leq_\alpha y$ *if and only if* $n \leq m$, *and* $(x_1, \ldots, x_n, \overbrace{\vee x_i, \ldots, \vee x_i}^{m-n}) \leq_\pi (y_1, \ldots, y_m)$
- $x \leq_\beta y$ *if and only if* $n \geq m$, *and* $(x_1, \ldots, x_n) \leq_\pi (y_1, \ldots, y_m, \overbrace{\wedge y_i, \ldots, \wedge y_i}^{n-m})$

where $\vee x_i$ *and* $\wedge y_i$ *stand for* $\max\{x_1, \ldots, x_n\}$ *and* $\min\{y_1, \ldots, y_m\}$*, respectively.*

Proposition 1 *The relations* $\leq_\pi, \leq_\alpha$ *and* $\leq_\beta$ *are orders on the set* E*.*

Remark 1 a) $x \leq_\pi y$ if and only if $x \leq_\alpha y$ and $x \leq_\beta y$.

b) Given $x = (x_1, \ldots, x_n) \in E$, we will denote $\underline{x} = (x_1, \ldots, x_n, \wedge x_i)$ and $\overline{x} = (x_1, \ldots, x_n, \vee x_i)$. Observe that $\underline{x} \leq_\beta x \leq_\alpha \overline{x}$.

Consider a multi-dimensional function $f : E \to [0,1]$. Observe that it can be viewed as a sequence $f = f_1, f_2, \ldots$ where, for each $n \geq 1$, $f_n : [0,1]^n \to [0,1]$ is the restriction of f to $[0,1]^n$.

Definition 2 *A function $f : E \to [0,1]$, $f = f_1, f_2, \ldots$, is a Multi-dimensional Aggregation Function (f is a MAF) if it satisfies the following conditions:*

1) Each f_n is continuous, non-decreasing and idempotent.

2) f is non-decreasing with respect to $\leq_\alpha$ and $\leq_\beta$:

a) $x \leq_\alpha y \Longrightarrow f(x) \leq f(y)$
b) $x \leq_\beta y \Longrightarrow f(x) \leq f(y)$

A useful characterization of MAFs is the following:

Proposition 2 *A multi-dimensional function $f = f_1, f_2, \ldots$ is a MAF if and only if*

1) $f_1 = id$.

2) Each f_n is continuous and non-decreasing.

3) For each $n \geq 1$ and for each $x \in [0,1]^n$,
$f_{n+1}(\underline{x}) \leq f_n(x) \leq f_{n+1}(\overline{x})$.

Example 1 The following elementary multi-dimensional functions are MAFs:

a) $f_n(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$,
$f_n(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$.

b) $f_n(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$,
$f_n(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdots x_n}$.

c) $f_n(x_1, \ldots, x_n) = x_1$, $f_n(x_1, \ldots, x_n) = x_n$.

Remark 2 Given a MAF f, then, for each $n \geq 1$ and $x \in [0,1]^n$,

$$\min\{x_1, \ldots, x_n\} \leq f(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}.$$

3 MULTI-DIMENSIONAL AGGREGATION FUNCTIONS GENERATED BY ITERATION

In this section we present a natural procedure of generating multi-dimensional aggregation functions from a bi-dimensional function.

Given a function $\varphi : [0,1]^2 \to [0,1]$, we can consider the multi-dimensional function $f = f_1, f_2, \ldots$ defined as:

1) $f_1 = id$, and

2) $f_n(x_1, \ldots, x_n) = \varphi(f_{n-1}(x_1, \ldots, x_{n-1}), x_n)$,
$\forall\, n \geq 2$.

When f is generated by φ, we will write $f =< \varphi >$.

It is interesting to observe that when $\varphi = T$ is a t-norm, then $f =< \varphi >$ is monotonic with respect to $\leq_\beta$ and when $\varphi = S$ is a t-conorm, then $f =< \varphi >$ is monotonic with respect to $\leq_\alpha$. Moreover, if f is generated by a t-norm T and it is non decreasing with respect to $\leq_\alpha$, then $T = \min$. Analogously, if f is generated by a t-conorm S and it is non decreasing with respect to $\leq_\beta$, then $S = \max$.

Proposition 3 *Given $\varphi : [0,1]^2 \to [0,1]$, let us consider the multi-dimensional function $f =< \varphi >$. Then f is a MAF if and only if φ is continuous, non-decreasing and idempotent.*

Example 2 a) The MAFs of the example 1a are generated, respectively, by $\varphi = \min$ and $\varphi = \max$.

b) The arithmetic and geometric means are not generated by iteration. In fact, the only MAF $f =< \varphi >$ with each f_n a quasi-linear arithmetic mean has the form:

$$\begin{aligned} f_n(x_1, \ldots, x_n) &= g^{-1}((1-\lambda)^{n-1} g(x_1) + \\ &+ \lambda(1-\lambda)^{n-2} g(x_2) + \cdots + \\ &+ \lambda(1-\lambda) g(x_{n-1}) + \lambda g(x_n)) \end{aligned}$$

where $\lambda \in [0,1]$ and g is a continuous, strictly monotonic function defined on $[0,1]$.

c) The functions of the example 1c can be obtained by taking $\lambda = 0$ or 1 and $g = id$ in the expression above.

Proposition 4 *Given a MAF $f =< \varphi >$, then f is symmetric (each f_n is symmetric) if and only if there exists $k \in [0,1]$ such that*

$$f_n(x_1, \ldots, x_n) = \begin{cases} \max\{x_1, \ldots, x_n\} & \textit{if } x_i \leq k \ \forall i \\ \min\{x_1, \ldots, x_n\} & \textit{if } x_i \geq k \ \forall i \\ k & \textit{otherwise} \end{cases}$$

for all n.

The proof of this proposition relies on the fact that if f is symmetric, then φ is associative and symmetric and hence, it is a (Max-Min)-operator, in the sense given in [4].

4 MULTI-DIMENSIONAL FUZZY MEASURES

For each $n \in \mathbb{N}$, let us consider the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

Definition 3 *A fuzzy measure on $\mathbb{N}_n$ is a set function $\mu_n : 2^{\mathbb{N}_n} \to [0,1]$ satisfying the following conditions:*

1) $\mu_n(\emptyset) = 0,\ \mu_n(\mathbb{N}_n) = 1.$

2) $A \subset B \Rightarrow \mu_n(A) \leq \mu_n(B) \quad \forall A, B \in 2^{\mathbb{N}_n}.$

From now on, given n numbers $x_1, \ldots, x_n$, $x_{(1)}, \ldots, x_{(n)}$ will denote the sequence arranged in increasing order: $x_{(1)} \leq \cdots \leq x_{(n)}$.

Definition 4 *Given a fuzzy measure μ, the n-dimensional discrete Sugeno integral with respect to μ is the function $S_\mu : [0,1]^n \to [0,1]$ defined by*

$$S_\mu(x_1, \ldots, x_n) = \bigvee_{i=1}^{n} (x_{(i)} \wedge \mu(A_{(i)}))$$

where $A_{(i)} = \{(i), (i+1), \ldots, (n)\}$.

Definition 5 *Given a fuzzy measure μ, the n-dimensional discrete Choquet integral with respect to μ is the function $C_\mu : [0,1]^n \to [0,1]$ defined by*

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)})\, \mu(A_{(i)})$$

with the same notations as above, and $x_{(0)} = 0$.

Definition 6 *A function $f : [0,1]^n \to [0,1]$ is called a discrete Sugeno (Choquet) integral if there exists a fuzzy measure μ on $\mathbb{N}_n$ such that $f = S_\mu$ ($f = C_\mu$).*

Remark 3 The Sugeno and Choquet integrals are idempotent, non-decreasing and continuous operators.

Different authors ([5]) have studied the problem of when a multi-dimensional OWA operator, or a multi-dimensional quasi-arithmetic weighted mean, $f = f_1, f_2, \ldots$ is a MAF. As we know ([2]), both the OWA operators ([7]) and the arithmetic means are discrete Choquet integrals (with respect to symmetric and additive fuzzy measures, respectively). Thus we propose now the problem of when a multi-dimensional discrete Choquet (or Sugeno) integral is a MAF. First of all, we must define such multi-dimensional functions.

Let us consider a sequence $\mu = \{\mu_n : n \geq 1\}$ of fuzzy measures, where each μ_n is defined on $\mathbb{N}_n$. Then we can consider the following two kinds of multi-dimensional operator $f : \bigcup_{n \geq 1} [0,1]^n \to [0,1]$:

- Multi-dimensional discrete Choquet integral:
  For each $n \geq 1$ and $(x_1, \ldots, x_n) \in [0,1]^n$,
  $f_n(x_1, \ldots, x_n) = C_{\mu_n}(x_1, \ldots, x_n)$.
- Multi-dimensional discrete Sugeno integral:
  For each $n \geq 1$ and $(x_1, \ldots, x_n) \in [0,1]^n$,
  $f_n(x_1, \ldots, x_n) = S_{\mu_n}(x_1, \ldots, x_n)$.

We would like to know when these operators are MAF. We already know that, for each $n \geq 1$, C_{μ_n} and S_{μ_n} are continuous, idempotent and non-decreasing. Then we must impose the non decreasingness with respect to the orders $\leq_\alpha$ and $\leq_\beta$.

Proposition 5 *Given $\mu = \mu_1, \mu_2, \ldots$ a sequence of fuzzy measures, let us consider the multi-dimensional function $f = f_1, f_2, \ldots$, where each f_n is a discrete Choquet integral with respect to μ_n. Then f is a MAF if and only if the sequence μ of fuzzy measures satisfies the inequalities*

$$\mu_{n+1}(A) \leq \mu_n(A) \leq \mu_{n+1}(A \cup \{n+1\})$$

$\forall\, n \geq 1$ and $A \in 2^{\mathbb{N}_n}$.

Proof. Suppose first that these two inequalities are satisfied. We prove the three conditions stated in the proposition 2. The first two conditions are obvious from the remark 4 and the third condition comes directly from the inequalities on μ. The proof of the converse is obtained by giving values 0 and 1 properly to the arguments of f_n and f_{n+1} in the condition 3 of the proposition 2.

It is interesting to observe that, when f is a multi-dimensional discrete Sugeno integral, we obtain the same result.

Remark 4 Observe that, when μ is a sequence of symmetric fuzzy measures (that is, f is a multi-dimensional OWA operator), this condition is equivalent to say that the weighting triangle $\bigwedge \omega_i^n$ associated to f is regular ([5]). An interesting study related to the regularity can be found in [1].

In the case of simple fuzzy measures, that is, $\mu(A) = 0$ or $1 \ \ \forall\, A \in 2^{\mathbb{N}}$, the inequalities of proposition 5 are equivalent to the two conditions: $\forall n$, $\forall A \subset \{1, \ldots, n\}$,

If $\mu_n(A) = 0 \Longrightarrow \mu_{n+1}(A) = 0$

If $\mu_n(A) = 1 \Longrightarrow \mu_{n+1}(A \cup \{n+1\}) = 1$

5 MULTI-DIMENSIONAL FUZZY MEASURE GENERATED BY A T-CONORM

Let S be a t-conorm and, for each $n \geq 2$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0,1]^n$ such that $S(\alpha_1, \ldots, \alpha_n) = 1$. We can define a fuzzy measure $\mu_{S,\alpha} : 2^{\mathbb{N}_n} \to [0,1]$ as

$$\mu_{S,\alpha}(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \alpha_i & \text{if } A = \{i\} \\ S(\alpha_{i_1}, \ldots, \alpha_{i_p}) & \text{if } A = \{i_1, \ldots, i_p\}, \\ & 2 \leq p \leq n \end{cases}$$

It is easy to proof the following result:

Proposition 6 $\mu_{S,\alpha}$ *is a fuzzy measure and, moreover,*

$$\max\{\mu_{S,\alpha}(A), \mu_{S,\alpha}(B)\} \leq \mu_{S,\alpha}(A \cup B) \leq S(\mu_{S,\alpha}(A), \mu_{S,\alpha}(B)),$$

$\forall\, A, B \in 2^{\mathbb{N}_n}$.

In the second inequality, the equality holds, $\mu_{S,\alpha}(A \cup B) = S(\mu_{S,\alpha}(A), \mu_{S,\alpha}(B)) \quad \forall\, A, B \in 2^{\mathbb{N}_n}$, *if and only if* $\alpha_1, \ldots, \alpha_n$ *are* S*-idempotent. In this case,* $\mu_{S,\alpha}(\{i_1, \ldots, i_p\}) = \max\{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$, $2 \leq p \leq n$.

These fuzzy measures have a strong multi-dimensional character. It is enough, in the definition, to consider a triangle of α's, $\bigwedge \alpha_i^n$, such that $\alpha_1^1 = 1$ and the t-conorm applied to the elements of each row is 1: $S(\alpha_1^n, \ldots, \alpha_n^n) = 1$, $\forall\, n \geq 2$. Then, for each $n \geq 1$, μ_n is the fuzzy measure defined as above. We call μ the sequence $\{\mu_n : n \geq 1\}$. Next we give a necessary condition and a sufficient condition for such a fuzzy measure to produce a multi-dimensional aggregation function.

Proposition 7 *a) In the conditions above, if* C_μ *is a multi-dimensional aggregation function, then* $\alpha_i^{n+1} \leq \alpha_i^n \leq S(\alpha_i^{n+1}, \alpha_{n+1}^{n+1})$ $\forall\, n \geq 1$, $\forall\, i = 1, \ldots, n$.

b) Suppose that the triangle $\bigwedge \alpha_i^n$ *satisfies that* $\alpha_i^{n+1} \leq \alpha_i^n \leq S(\alpha_i^{n+1}, \alpha_{n+1}^{n+1})$ $\forall\, n \geq 1$, $\forall\, i = 1, \ldots, n$ *and the elements* α_n^n *are* S*-idempotent for all* $n \geq 1$*. Then* C_μ *is a multi-dimensional aggregation function.*

Remark 5 The converse is not true, in general, in both results. In effect, in the first case, consider the t-conorm $S(x,y) = \min\{x+y, 1\}$ and the triangle with third row $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$ and fourth row $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8})$. Similarly, in the second case, consider the same t-conorm and the triangle with third row $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$ and fourth row $(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3})$.

Observe that the condition of the elements α_n^n, $n \geq 1$, being S-idempotents is very strong. A usual multi-dimensional aggregation function like the arithmetic mean $f(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$ comes from the fuzzy measure $\mu(A) = \frac{|A|}{n}$, that is, $S(x,y) = \min\{x+y, 1\}$ and $\alpha_i^n = \frac{1}{n}$ $\forall i = 1, \ldots, n$. We give next an equivalent condition for the case of additive fuzzy measures which does not impose idempotency of any element of the triangle.

We know that, if μ_n is an additive fuzzy measure on $\mathbb{N}_n$, then the discrete Choquet integral with respect to μ_n, C_{μ_n}, is a weighted arithmetic mean, with weights $\omega_i^n = \mu_n(\{i\})$. We also know that such a measure must be generated by the t-conorm $S(x,y) = \min\{x+y, 1\}$ and $(\alpha_1, \ldots, \alpha_n)$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

Proposition 8 *Let us consider a sequence* μ *such that every fuzzy measure* μ_n $(n \geq 1)$ *is additive. Then* C_μ *is a multi-dimensional aggregation function if and only if the triangle* $\bigwedge \alpha_i^n$ *satisfies that* $\alpha_i^{n+1} \leq \alpha_i^n$ $\forall\, n \geq 1$, $\forall\, i = 1, \ldots, n$.

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References

[1] M. Carbonell, M. Mas, G. Mayor. On a class of Monotonic Extended OWA Operators. *FUZZ-IEEE'97*, pp. 1695-1700. Barcelona, 1997.

[2] M. Grabisch. Fuzzy integral in multicriteria decision making. *Fuzzy Sets and Systems, 69*, pp. 279-298, 1995.

[3] M. Grabisch, H. Nguyen, E. Walker. Fundamentals of Uncertainty Calculi, with Applications to Fuzzy Inference. *Kluwer Academic*, 1995.

[4] M. Mas, G. Mayor, J. Torrens. t-operators. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems.* Vol. 7, 1, pp 31-50, 1999.

[5] G. Mayor, T. Calvo. On Extended Aggregation Functions. *IFSA'97*, pp. 281-285. Praga, 1997.

[6] M. Sugeno. Theory of fuzzy integrals and its applications. PhD thesis, Tokyo Institute of Technology, 1974.

[7] R.R. Yager. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Trans. on SMC, 18*, pp. 183-190, 1988.