GENERALISED QUALITATIVE UTILITY FUNCTIONS FOR REPRESENTING PARTIAL PREFERENCES RELATIONS

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Summary

Representational issues of preferences in the framework of a possibilistic (ordinal) decision model under uncertainty were introduced by Dubois and Prade quite recently. In this framework, (finite) linear uncertainty and preference scales are assumed, and decisions can be ranked according to their expected utility in terms of Sugeno integrals. In this paper we generalise the model by allowing (i) to measure uncertainty and preferences on non linear lattices, and (ii) to formulate the utility expectations in terms of generalized Sugeno integrals where t-norms and t-conorms play a role. For these generalised utility functions we provide axiomatic characterisations. Finally, we propose how to extend the utility functions to cope with belief states that may be partially inconsistent.

1 Introduction

The representational issues of preferences in the framework of a possibilistic (qualitative / ordinal) decision model under uncertainty, were originally introduced a few years ago by Dubois and Prade [4], and more recently linked to case-based decision problem in [3]. They proposed a qualitative counterpart to Von Neumann and Morgenstern's Expected Utility Theory [5]. In this approach, the uncertainty is assumed to be of possibilistic nature, i.e. belief states are represented by normalised possibility distributions π, and make use of finite commensurate ordinal preference and uncertainty scales. However, sometimes we may be faced with case-based decision problems where similarity may take values that are incomparable, Decision Maker's preferences may be partial, etc.. In many applications we have to measure degrees of similarity or preferences in partially ordered sets, so this may be an interesting subject which we will investigate in a possibilistic context. Hence, to cope with these situations, as a first step, an extension of the above approach is proposed in [6], where both preferences and uncertainty are graded on finite distributive lattices, $(U, \wedge, \vee, 0, 1, n_U)$ and $(V, \wedge, \vee, 0, 1, n_V)$ respectively, that are commensurate. Axiomatic settings for characterising two qualitative utilities, a pessimistic and an optimistic one are provided. Given $u : X \to U$ a preference function that assigns to each consequence of X a preference level of U, and $h : V \to U$ a {0,1}-homomorphism relating both lattices V and U. Let n be the reversing homomorphism $n : V \to U$ defined as $n(\lambda) = n_U(h(\lambda))$, also verifying $n(0) = 1$, $n(1) = 0$. For any normalised π, i.e. $\pi : X \to V$ s.t. exists $x|\ \pi(x) = 1$, we considered the qualitative utility functions:

$$QU^-(\pi|u) = \bigwedge_{x\in X}(n(\pi(x)) \vee u(x)),$$
$$QU^+(\pi|u) = \bigvee_{x\in X}(h(\pi(x)) \wedge u(x)).$$

QU^- is a pessimistic criterion that looks for those π's which, at some extent, make hardly plausible all the bad consequences. On the contrary, QU^+ is an optimistic criterion which looks for those π's that, also to some extent, makes plausible some of the good consequences.

In this paper we consider other alternatives to obtain generalised utility functions GQU^- and GQU^+, namely the use of a t-norm instead of $\wedge$ in QU^- and a t-conorm instead of $\vee$ in QU^+. We will show that, under some conditions, GQU^- and GQU^+ are indeed "utility" functions in the set of normalised possibility distributions, in the sense that they preserve the preference ordering and the "natural operation" of possibilistic mixture used to combine possibilistic lotteries or distributions. In a recent paper [7], we have proposed an extension of the possibilistic decision model to deal with non-normalised possibility distributions i.e. distributions accounting for partially belief states. Following this proposal, now we also take into account this ability in the generalised model and show its usefulness to provide elements for a qualitative case-based decision methodology.

In the following section we provide a background on lattices, while in section 3 the context of our work is given. In section 4 we propose two axiomatic settings for characterising both pessimistic and optimistic generalised qualitative utilities requiring finite distributive and commensurate lattices for assessing uncertainty and preferences; also including an extension that lets us make decisions in contexts in which possibly partially inconsistent belief states are involved.

2 Lattices and Possibility Distributions

In this preliminary section we introduce some notation, definitions and results related to lattices (for more details see [2]) and possibility distributions that we will use in the rest of the paper.

Let L be a set and $\wedge$ and $\vee$ two binary operations on L. $(L, \wedge, \vee)$ is a *lattice* if $\wedge, \vee$ are associative, commutative, satisfy idempotency and the absorption laws. The induced order in the lattice is: $x \leq y$ iff $x \wedge y = x$. $(L, \wedge, \vee, n_L, 0, 1)$ will denote a *bounded lattice with involution*, i.e. $0, 1 \in L$, $0 \leq x \leq 1$, for all $x \in L$, and $\mathrm{n}_L : L \to L$ a decreasing function s.t. $n_L(n_L(x)) = x$. Note that n_L satisfies that $n_L(0) = 1, n_L(1) = 0, n_L(x \wedge y) = n_L(x) \vee n_L(y)$ and $n_L(x \vee y) = n_L(x) \wedge n_L(y)$.

Given a partially pre-ordered set $(L, \leq)$, i.e. $\leq$ is reflexive and transitive, the associated indifference and incomparability relations are defined as:

$$x \sim y \iff x \leq y \text{ and } y \leq x.$$
$$x <> y \iff x \not\leq y \text{ and } y \not\leq x.$$

Let $L/\sim$ denote *the quotient set* w.r.t. the equivalence relation $\sim$ and let us denote also by $\leq$ the ordering induced by L into $L/\sim$. Then, we shall call $(L, \leq)$ a *pre-lattice* iff $(L/\sim, \leq)$ is a bounded lattice with the meet and joint operations defined from $\leq$. Now, given a lattice $(L, \wedge, \vee, 0, 1)$, *a t-norm* (t-conorm) operation $\top(\perp)$ on L is any non-decreasing, associative and commutative binary operation on L, verifying $\lambda\top 0 = 0$ and $\lambda \top 1 = \lambda$ ($\lambda \perp 0 = \lambda$ and $\lambda \perp 1 = 1$, resp.) for all $\lambda \in L$. The *residuum of* $\top$ is defined as

$$\mathrm{I(a, c)} = \bigvee\{b \in L \mid \top(\mathrm{a, b}) \leq \mathrm{c}\}.$$

$(L, \wedge, \vee, \top, \mathrm{I}, 0, 1)$ is a *residuated lattice* if $(L, \wedge, \vee, 0, 1)$ is a lattice and $(\top, \mathrm{I})$ is an adjoint pair, i.e. if

- $(L, \top, 1)$ is a commutative semigroup with unit element 1.
- $\forall a, b, c \in L$ $(a \top b) \leq c$ iff $a \leq \mathrm{I}(b, c)$.

Finally, one can check that if we have a finite lattice $(L, \wedge, \vee, 0, 1)$ and a t-norm $\top$ on L, then $\top$ distributes over the lattice joint operation (i.e. $(a \vee b)\top c = (a \top c) \vee (b \top c), \forall a, b, c \in L$) iff $(L, \wedge, \vee, \top, \mathrm{I}, 0, 1)$ is a residuated lattice.

The *lattice of uncertainty values* $(V, \wedge, \vee, 0, 1, n_V, \top)$ will be a finite distributive lattice with two additional operations: an involution n_V and a t-norm $\top$ on V. The *lattice of preference values* $(U, \leq_U, 0, 1, n_U)$ will be a finite distributive lattice with involution. We will denote by $Pi(X)$ the set of *consistent possibility distributions on X over V*, i.e.

$$Pi(X) = \{\pi : X \to V \mid \textstyle\bigvee_{x \in X} \pi(x) = 1\}.$$

The pointwise ordering on $Pi(X)$ is then defined as $\pi \leq \pi$' iff $\forall x \in X$ $\pi(x) \leq_V \pi'(x)$, with $\leq_V$ the order induced by $\wedge$ in V. We will be interested in a subset of $Pi(X)$, the set of *normalised* possibility distributions, i.e. the set

$$Pi^*(X) = \{\pi \in Pi(X) \mid \exists x \text{ s.t. } \pi(x) = 1\}.$$

For the sake of simplicity, we shall use A for denoting both both a subset $A \subseteq X$ and the normalised possibility distribution on X such that $\pi(x) = 1$ if $x \in A$ and $\pi(x) = 0$ otherwise. Hence, we can consider X as included in $Pi^*(X)$.

3 Generalised Utility functions for $\vee$-Mixtures

As it has been mentioned, QU^- and QU^+ are "utility" functions in $Pi^*(X)$, in the sense that they preserve the preference ordering and the maximin combination of possibilistic mixture. Now, we analyse the conditions required to guarantee that the functions GQU^- and GQU^+ preserve a possibilistic mixture. Instead of applying maximin combination of possibility distributions, we consider other mixtures involving t-conorms and t-norms. For each t-norm $\top$ and conorm $\perp$ on V, we will be interested in $\perp$-$\top$ mixtures that combine two possibility distributions π_1 and π_2 into a new one, denoted $M_{\top,\perp}(\pi_1, \pi_2, \lambda, \mu)$, with $\lambda, \mu \in V$ and $\lambda \perp \mu = 1$, defined as

$$M_{\top,\perp}(\pi_1, \pi_2, \lambda, \mu)(x) = (\lambda \top \pi_1(x)) \perp (\mu \top \pi_2(x)).$$

We require these mixtures to satisfy reduction of lotteries, that is:

$$M_{\top,\perp}(M_{\top,\perp}(\pi_1, \pi_2, \lambda_1, \lambda_2), M_{\top,\perp}(\pi_1, \pi_2, \mu_1, \mu_2), \alpha, \beta)$$
$$= M_{\top,\perp}((\pi_1, \pi_2, (\alpha \top \lambda_1) \perp (\beta \top \mu_1), (\alpha \top \lambda_2) \perp (\beta \top \mu_2)).$$

Hence, we need to satisfy $(a \top c) \perp (b \top c) = c \top (a \perp b)$. Therefore, we have to restrict to $\vee - \top$ mixtures [1], moreover, we have to require $(V, \wedge, \vee, \top, \mathrm{I}, 0, 1)$ to be a residuated lattice, hence forth V will be referred as a residuated lattice. So, for each t-norm $\top$ on V, we may consider the so-called Possibilistic Mixture. In order to have a closed operation on $Pi^*(X)$, the mixture operation is restricted to $Pi^*(X)$ requiring the scalars to satisfy an additional condition $\lambda = 1$ or $\mu = 1$. As it is said, we will consider here other alternative for modelling implication, we define

$$(v \Rightarrow u) = n(v \top z)$$

with $n(z) = u$, $\top$ a t-norm on V and $n = n_U \circ h$, being n_U the involution in U, and $h : V \to U$ an order preserving function, such that $h(0) = 0, h(1) = 1$. Hence, given a preference function $u : X \to U$ that assigns to each consequence of X a preference level of U, for a pessimistic behaviour we propose

$$GQU^{-}(\pi|u) = \bigwedge_{x\in X} n(\pi(x)\top\lambda_x).$$

being λ_x s.t. $n(\lambda_x) = u(x)$. To guarantee the correctness of the above definition of implication we require h to satisfy the following coherence condition w.r.t. $\top$,

$$h(\lambda) = h(\mu) \Rightarrow h(\alpha\top\lambda) = h(\alpha\top\mu) \quad \forall\alpha,\lambda,\mu \in V.$$

Notice that either when $\top = \wedge$ or when h is injective this condition is satisfied. If h is coherent w.r.t. $\top$, so is n. Observe that the implication proposed may be seen like a generalisation of a S-implication. Actually, when h is injective, h satisfies coherence, then

$$(v \Rightarrow u) = n(v\top z) = n(v)\bot_{n,\top} u,$$

being $\bot_{n,\top}$ the conorm in U defined as $(n(\lambda)\bot_{n,\top} n(\lambda')) = n(\lambda\top\lambda')$. That is, $(v \Rightarrow u)$ is a S-implication w.r.t. the conorm $\bot_{n,\top}$. Instead, for an optimistic behaviour we consider the t-norm as the conjunction, so we propose

$$GQU^{+}(\pi|u) = \bigvee_{x\in X} h(\pi(x)\top\mu_x)$$

being μ_x s.t. $u(x) = h(\mu_x)$. Observe that if h is join-preserving and as V is a residuated lattice with involution, then GQU^- and GQU^+ preserves the possibilistic mixture in the sense that the following expressions hold,

$$GQU^{-}(M_\top(\pi_1,\pi_2,\lambda,\mu)|u)(x) = n(\lambda\top\delta_1) \wedge n(\mu\top\delta_2),$$
$$GQU^{+}(M_\top(\pi_1,\pi_2,\lambda,\mu)|u)(x) = h(\lambda\top\gamma_1) \vee h(\mu\top\gamma_2),$$

with $n(\delta_j) = GQU^-(\pi_j|u)$, $h(\gamma_j) = GQU^+(\pi_j|u)$,

4 Representation of Generalised Qualitative Utilities

First, we propose a set of axioms to characterise pessimistic and optimistic qualitative utilities for normalised possibility distributions. Then, in order to may apply the model in contexts involving partially inconsistent belief states, we introduce utility functions for evaluating possibly non normalised distributions and their corresponding axiomatic characterisations.

Proposition 1 *Let $(Pi^*(X), \sqsubseteq)$, satisfying*

- *A1: $(Pi^*(X), \sqsubseteq)$ is a pre-lattice.*
- *A2(uncertainty aversion): if $\pi \leq \pi' \Rightarrow \pi \sqsupseteq \pi'$.*

Then

a) The maximal elements of $(Pi^*(X), \sqsubseteq)$ are equivalent.

b) The maximal elements of $(X, \sqsubseteq)$ are equivalent, and they are equivalent to the maximal ones of $(Pi^*(X), \sqsubseteq)$.

Axiomatic setting: Let $AXP_\top$ be the following set of axioms on $(Pi^*(X), \sqsubseteq, M_\top)$,

- A1: $(Pi^*(X), \sqsubseteq)$ is a pre-lattice.
- A2 (uncertainty aversion): if $\pi \leq \pi' \Rightarrow \pi \sqsupseteq \pi'$.
- A3 (independence): $\pi_1 \sim \pi_2 \Rightarrow M_\top(\pi_1,\pi,\lambda,\mu) \sim M_\top(\pi_2,\pi,\lambda,\mu)$.

Let $\overline{\pi}$ be a maximal element of $(Pi^*(X), \sqsubseteq)$, then $\pi_\lambda = M_\top(\overline{\pi}, X, 1, \lambda)$.

- A4: if $\pi_\lambda \sqsubseteq \pi'_\lambda \Rightarrow \pi_{n_V(\lambda)} \sqsupseteq \pi_{n_V(\lambda')}$,
- A5: if $\lambda <> \lambda' \Rightarrow \pi_\lambda \sqsubset\sqsupset \pi_{\lambda}$'.
- A6: $\forall\pi \in Pi^*(X), \exists\lambda \in V$ s.t. $\pi \sim M_\top(\overline{\pi}, X, 1, \lambda)$.

Lemma 2 *Let $(U, \leq_U, 0, 1, n_U)$ and $(V, \wedge, \vee, \top, I, 0, 1)$ be two lattices with involution, $h : V \to U$ an onto join-preserving mapping satisfying coherence w.r.t. $\top$, and $u : X \to U$. If $(GQU^-)^{-1}(1) \neq \emptyset$[1] and $(GQU^-)^{-1}(0) \neq \emptyset$,then*

- *there exists $x \in X$ s.t. $u(x) = 1, \bigwedge_{x\in X} u(x) = 0$*
- *GQU^- is onto*

Lemma 3 *Let $h : V \to U$ be an onto non decreasing function also satisfying that if $\lambda <> \lambda$' then $h(\lambda) <>_U h(\lambda$'). Then, h is a preserving epimorphism.*

Now, let $\preceq_{GQU}$ $-$ be the preference ordering on $Pi^*(X)$ induced by GQU^-, i.e.

$$\pi \preceq_{GQU-} \pi\text{' iff } GQU^-(\pi|u) \leq_U GQU^-(\pi'|u).$$

Theorem 4 *A preference relation $(Pi^*(X), \sqsubseteq, M_\top)$ satisfies axioms $AXP_\top$ iff there exist*

- *a utility finite distributive lattice $(U, \wedge, \vee, n_U, 0,1)$,*
- *a preference function $u : X \to U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x\in X} u(x) = 0$,*
- *an onto join-morphism $h : V \to U$, i.e. $h(\lambda \vee \lambda') = h(\lambda) \vee h(\lambda$'), s.t. $h(0) = 1, h(1) = 1$, satisfying coherence w.r.t. $\top$, $n_U \circ h \circ n_V = h$, and also satisfying*

  if $\lambda <> \lambda'$ then $h(\lambda) <>_U h(\lambda$'),

in such a way that it holds:

$$\pi' \sqsubseteq \pi \textit{ iff } GQU^-(\pi|\ u) \leq_U GQU^-(\pi'|\ u).$$

In order to represent an optimistic preference criterion, we consider now the distribution π_λ defined as $\pi_\lambda = M_\top(X, \underline{\pi}, \lambda, 1)$, where $\underline{\pi}$ is minimal of $(Pi^*(X), \sqsubseteq)$, and we have to change the uncertainty aversion axiom A2 by an uncertainty-prone postulate

- A2$^+$: if $\pi \leq \pi$' then $\pi \sqsubseteq \pi$'

and to modify the continuity axiom A6 into

- A6$^+$: $\forall\pi \in Pi^*(X)\exists\lambda \in V$ s.t. $\pi \sim M_\top(X, \underline{\pi}, \lambda,$

For an optimistic behaviour, we may consider $\preceq_{GQU^+}$ the preference ordering on $Pi^*(X)$ induced by GQU^+. The corresponding representation theorem is analogous to the pessimistic case and it is omitted.

Representation of Utilities for Non Normalised Distributions

Applying these models to non normalised distributions may conclude in unsatisfactory effects like that the

[1]For a simpler notation we will omit u when there is no confusion about it.

pessimistic utility may result higher than the optimistic utility. In order to avoid some problems involving non normalised distributions, we provide now the corresponding extension of our initial proposal. First, let us introduce the concepts of normalisation and height of a distribution. Let $\mathcal{H}$ be the height of a distribution, i.e. $\mathcal{H}(\pi) = \bigvee_{x \in X} \pi(x)$, for each distribution we consider the subset of consequences with maximal plausibility

$$X_{\mathcal{H}(\pi)} = \{x \in X \mid \forall y \in X \ \pi(y) \not> \pi(x)\}.$$

Define $\mathcal{N}(\pi)$ the normalisation of π, i.e. the normalised distribution

$$\mathcal{N}(\pi) = \begin{cases} 1 & if \quad x \in X_{\mathcal{H}(\pi)} \\ \pi(x) & otherwise \end{cases}$$

and consider the extension of the set of possibilistic lotteries to the set $Pi^{ex}(X)$ of non necessarily normalised distributions on V. Now, we propose the qualitative (or ordinal) utility functions on $Pi^{ex}(X)$,

$$\begin{aligned} \underline{GQU^-}(\pi|u) &= GQU^-(\mathcal{N}(\pi)|u) \wedge n(n_V(\mathcal{H}(\pi))) \\ \underline{GQU^+}(\pi|u) &= GQU^+(\mathcal{N}(\pi)|u) \vee h(n_V(\mathcal{H}(\pi))). \end{aligned}$$

Let $\sqsubseteq_{ex}$ be a preference relation in $Pi^{ex}(X)$. We will denote by $\sqsubseteq$ its restriction to $Pi^*(X)$, $\sim_{ex}$ and $\sim$ the corresponding indifference relations. In order to characterise the preference orderings induced by $\underline{GQU^-}$, we extend the axiom set $AXP_\top$, defined on $\overline{(Pi^*(X), \sqsubseteq, M_\top)}$, with A7P:

- $\forall \pi \in \ Pi^{ex}(X) \pi \sim_{ex} M_\top(\mathcal{N}(\pi), X, 1, n_V(\mathcal{H}(\pi)))$.

The intuitive idea behind this axiom is that we make a non-normalised possibilistic lottery π indifferent to the corresponding normalised lottery $\mathcal{N}(\pi)$, provided that it is modified by a uniform uncertainty level corresponding to the inconsistency degree of π.
We say that $\sqsubseteq_{ex}$ on $Pi^{ex}(X)$ satisfies axiom set $\underline{AXP}_\top = AXP_\top \cup \{A7P\}$ iff $\sqsubseteq$ satisfies $AXP_\top$ and $\sqsubseteq_{ex}$ also satisfies A7P.

Theorem 5 *A preference relation $\sqsubseteq_{ex}$ on $Pi^{ex}(X)$ satisfies axiom set $\underline{AXP}_\top$ iff there exist*

- *a utility finite distributive lattice $(U, \wedge, \vee, n_U,$ 0, 1) with involution n_U,*
- *a preference function $u : X \to U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x \in X} u(x) = 0$,*
- *a preserving epimorphism $h : V \to U$ s.t. $h(0) = 0$, $h(1) = 1$, $n_U \circ h \circ n_V = h$, satisfying coherence w.r.t. $\top$, and if $\lambda <> \lambda'$ then $h(\lambda) <>_U h(\lambda')$,*

in such a way that it holds:

$$\pi' \sqsubseteq_{ex} \pi \ iff \ \underline{GQU^-}(\pi') \leq_U \underline{GQU^-}(\pi).$$

In a similar way, we characterise the "optimistic" utility on $Pi^{ex}(X)$, using the set $AXP^+_\top$ plus A7P.

Remark 1 . *Instead of using the involution n_V in the definition of $\underline{GQU^-}$ and $\underline{GQU^+}$, one could simply use a more general function $F : V \to V$ s.t. $F(1) = 0$,*

$$\begin{aligned} \underline{GQU_F}^-(\pi) = GQU^-(\mathcal{N}(\pi)) \wedge n(F(\mathcal{H}(\pi))) \\ \underline{GQU_F}^+(\pi) = GQU^+(\mathcal{N}(\pi)) \vee h(F(\mathcal{H}(\pi))). \end{aligned}$$

In that case, given such a function F, it is not difficult to show that Theorem 5 is still valid provided that we replace axiom A7P by an analogous one:

- $\forall \pi \in \ Pi^{ex}(X) \ \pi \sim_{ex} M_\top(\mathcal{N}(\pi), X, 1, F(\mathcal{H}(\pi)))$

5 Conclusions

We have been concerned with representational aspects of preference relations in the framework of a possibilistic (qualitative / ordinal) decision model under uncertainty. The model measures degrees of uncertainty (similarity) and preferences on partially ordered sets (finite lattices). Here, we assumed the availability of t-norms operators as well as the usual $\wedge, \vee$ ones. This allowed us to apply non purely ordinal expressions in the utility functions involved. Then, we extended the model to cope with decision problems where the belief state may be partially inconsistent, like those involved in case-based decision problem. This enables the model to rank, possibly non-normalised, possibility distributions on the set of decision consequences.

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