TRIANGULAR NORMS ON THE REAL UNIT SQUARE

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Summary

T-norms with vector arguments are introduced and investigated. Direct products of t-norms on the unit interval are characterized. T-norms without zero divisors, pseudo-Archimedean t-norms and cancellative t-norms are discussed. Some open problems are stated.

Keywords: cancellation law; direct product; pseudo-Archimedean property; t-norm; unit square.

1 INTRODUCTION

T-norms were introduced by Schweizer and Sklar (see e.g. [16]) in the framework of probabilistic metric spaces and are based on a notion used by Menger in order to extend the triangle inequality in the definition of metric spaces towards probabilistic metric spaces [13]. They have been used extensively for defining the intersection of fuzzy sets, and for modelling the logical 'and' in fuzzy logic. Continuous t-norms on the real unit interval have been completely characterized as ordinal sums of continuous Archimedean t-norms [11]. The structure of non-continuous t-norms is not known yet [10].

Due to the close connection between fuzzy set theory and order theory (see e.g. [8]), several authors have studied or worked with t-norms on bounded posets (see e.g. [3, 4, 5, 6, 9, 12, 14, 15] and many others). In this paper, we continue our study of t-norms on product lattices and focus on the particular case of the (real) unit hypercube $[0,1]^n$, i.e. we consider t-norms acting on vectors. From a practical point of view, the unit hypercube is the ultimate working environment: at the same time it allows for incomparability and also offers the possibility to draw back on the underlying real line.

2 T-NORMS ON $[0,1]^2$

T-norms can be introduced without any problem in a partially ordered framework. A t-norm T on a bounded poset $\mathbb{P} = (P, \leq, 0, 1)$ is defined as a commutative and associative increasing binary operation on P with neutral element 1. The partial order relation of $\mathbb{P}$ can be extended in a natural way to a partial order relation on the class of t-norms on it. For two t-norms T_1 and T_2 on $\mathbb{P}$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall (x,y) \in P^2)(T_1(x,y) \leq T_2(x,y)).$$

We are now interested in the particular case of the (real) unit hypercube $([0,1]^n, \leq)$. For the sake of simplicity, we will write all results for $n = 2$, i.e. we will consider the (real) unit square $([0,1]^2, \leq)$ equipped with the product ordering $\leq$ defined by

$$(x_1, y_1) \leq (x_2, y_2) \;\Leftrightarrow\; x_1 \leq x_2 \wedge y_1 \leq y_2.$$

The unit square is a bounded lattice with smallest element $(0,0)$, greatest element $(1,1)$ and meet operation $\frown$ and join operation $\smile$ defined by

$$(x_1, y_1) \frown (x_2, y_2) = (\min(x_1, x_2), \min(y_1, y_2))$$
$$(x_1, y_1) \smile (x_2, y_2) = (\max(x_1, x_2), \max(y_1, y_2)).$$

The smallest t-norm T_W on $[0,1]^2$ is given by

$$T_W((x_1, y_1), (x_2, y_2)) = \begin{cases} (x_1, y_1) & \text{, if } (x_2, y_2) = (1,1) \\ (x_2, y_2) & \text{, if } (x_1, y_1) = (1,1) \\ (0,0) & \text{, elsewhere} \end{cases}$$

and the greatest t-norm T_M is given by the meet operation $\frown$, i.e. for any t-norm T on $[0,1]^2$ it holds that $T_W \leq T \leq T_M$.

3 DIRECT PRODUCTS OF T-NORMS

One easily verifies that for two t-norms T_1 and T_2 on $[0,1]$, the *direct product* $T_1 \times T_2$ defined by

$$T_1 \times T_2((x_1,y_1),(x_2,y_2)) = (T_1(x_1,x_2),T_2(y_1,y_2))$$

is a t-norm on $[0,1]^2$. Clearly, the smallest t-norm T_W is not a direct product of t-norms on $[0,1]$. Recall the following characterization of direct products.

Theorem 3.1 [4] *Consider a t-norm T on $[0,1]^2$, then the following statements are equivalent:*

(i) *T is the direct product of two t-norms T_1 and T_2 on $[0,1]$;*

(ii) *the partial mappings of T are meet-morphisms;*

(iii) *the partial mappings of T are join-morphisms.*

The foregoing theorem is quite important. It implies that in a residual framework, i.e. when the partial mappings of the t-norm are assumed to be supremum-morphisms, only direct products can be used.

4 BASIC PROPERTIES OF T-NORMS

In [4], we have introduced the sets of idempotent elements, zero divisors and nilpotent elements associated with an arbitrary t-norm T on a bounded poset $\mathbb{P} = (P,\leq,0,1)$:

(i) An element $x \in P$ is called an *idempotent element* of T if $T(x,x) = x$. The set of idempotent elements of T is denoted by $I(T)$; 0 and 1 are called trivial idempotent elements.

(ii) An element $x \in P$ is called a *zero divisor* of T if there exists $y \in P$ such that $\mathcal{L}(\{x,y\}) \neq \{0\}$ and $T(x,y) = 0$. The set of zero divisors of T is denoted by $Z(T)$; if $Z(T) = \emptyset$ then T is called a t-norm without zero divisors.

(iii) An element $x \in P \setminus \{0\}$ is called a *nilpotent element* of T if there exists $n \in \mathbb{N}$ such that $x^{(n)_T} = 0$. The set of nilpotent elements of T is denoted by $N(T)$.

Note that the notation $\mathcal{L}(A)$ stands for the set of lower bounds of the subset A of P. The notation $x^{(n)_T}$ stands for $T(x,\ldots,x)$ (n times argument x). By convention, we have that $x^{(1)_T} = x$.

In the case of the unit square, it holds that (x_1,y_1) is a zero divisor of a t-norm T on $[0,1]^2$ if and only if there exists (x_2,y_2) such that $(x_1,y_1) \frown (x_2,y_2) \neq (0,0)$ and $T((x_1,y_1),(x_2,y_2)) = (0,0)$. Note that since $T \leq T_M$, $(x_1,y_1) \frown (x_2,y_2) = (0,0)$ always implies that $T((x_1,y_1),(x_2,y_2)) = (0,0)$. Hence, such elements (x_2,y_2) cannot influence the decision whether (x_1,y_1) is to be considered as a zero divisor.

Consider two t-norms T_1 and T_2 on $[0,1]$ and their direct product $T_1 \times T_2$ on $[0,1]^2$. Then the following equalities hold:

$$I(T_1 \times T_2) = I(T_1) \times I(T_2)$$

$$N(T_1 \times T_2) = ((N(T_1) \cup \{0\}) \times (N(T_2) \cup \{0\})) \setminus \{(0,0)\}$$

$$Z(T_1 \times T_2) = (Z(T_1) \times [0,1]) \cup ([0,1] \times Z(T_2)).$$

In particular, the first equality above implies that the Cartesian product of the sets of trivial idempotent elements is included in the set of idempotent elements of any direct product of t-norms. Hence, the direct product $T_1 \times T_2$ not only has $(0,0)$ and $(1,1)$ as idempotent elements, but also $(0,1)$ and $(1,0)$. The third equality implies that the direct product $T_1 \times T_2$ is a t-norm without zero divisors if and only if both T_1 and T_2 are t-norms without zero divisors.

5 PSEUDO-ARCHIMEDEAN T-NORMS

The definition of the Archimedean property of binary operators on a bounded lattice (see e.g. [1]) can be applied to t-norms on a bounded poset as well. A t-norm T on a bounded poset $\mathbb{P} = (P,\leq,0,1)$ is called *Archimedean* if for any $(x,y) \in P^2$ it holds that

$$(\forall n \in \mathbb{N})(x^{(n)_T} \geq y) \Rightarrow (x = 1 \vee y = 0).$$

Clearly, a t-norm T on $[0,1]$ is Archimedean if and only if

$$(\forall (x,y) \in]0,1[^2)(\exists n \in \mathbb{N})(x^{(n)_T} < y).$$

One easily verifies that a t-norm T on a bounded poset $\mathbb{P} = (P,\leq,0,1)$ is Archimedean if and only if

$$(\forall x \in P \setminus \{1\})(\mathcal{L}(\{x^{(n)_T} \mid n \in \mathbb{N}\}) = \{0\}).$$

Of course, we would like to see the Archimedean property preserved under direct products. Consider the direct product $T = T_1 \times T_2$ of two t-norms T_1 and T_2 on $[0,1]$. As mentioned above, $(0,1)$ and $(1,0)$ are both idempotent elements of T. Hence, for any $(x,1) \in [0,1]^2$ the monotonicity of T implies that $(0,1) \in \mathcal{L}(\{(x,1)^{(n)_T} \mid n \in \mathbb{N}\})$. Similarly, for any $(1,y) \in [0,1]^2$ it holds that $(1,0) \in \mathcal{L}(\{(1,y)^{(n)_T} \mid n \in \mathbb{N}\})$. Therefore, elements of the type $(x,1)$ or $(1,y)$ cannot be nilpotent elements and T cannot be Archimedean. In order to avoid this undesirable situation, we have modified the definition of the

Archimedean property of t-norms on $[0,1]^2$. It is clear that we have to exclude from our considerations the elements greater than or equal to one of the idempotent elements $(0,1)$ and $(1,0)$. More formally, this is the set

$$U = (\{1\} \times [0,1]) \cup ([0,1] \times \{1\}).$$

Definition 5.1 *A t-norm T on $[0,1]^2$ is called pseudo-Archimedean if for any $(x,y) \in [0,1]^2 \times [0,1]^2$ it holds that*

$$(\forall n \in \mathbb{N})(x^{(n)_T} \geq y) \Rightarrow (x \in U \vee y = (0,0)).$$

Proposition 5.1 *Consider two t-norms T_1 and T_2 on $[0,1]$, then the direct product $T_1 \times T_2$ is pseudo-Archimedean if and only if T_1 and T_2 are Archimedean.*

6 THE CANCELLATION LAW

In a similar way, the cancellation law can be modified in order to guarantee that the direct product of cancellative t-norms is again cancellative. Recall that a t-norm T on $[0,1]$ is called *cancellative* if for any $x \neq 0$ and any $(y,z) \in [0,1]^2$ the equality $T(x,y) = T(x,z)$ implies $y = z$. In this case, a t-norm T is cancellative if and only if its partial mappings $T(x,\cdot)$, $x \neq 0$, are strictly increasing.

As mentioned above, it holds for any t-norm T on $[0,1]^2$ that $T((x_1,y_1),(x_2,y_2)) = (0,0)$ whenever $(x_1,y_1) \frown (x_2,y_2) = (0,0)$. This implies that we have to exclude from our considerations those elements (x_1,y_1) for which there exists $(x_2,y_2) \neq (0,0)$ such that $(x_1,y_1) \frown (x_2,y_2) = (0,0)$. More formally, this is the set

$$Z = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}).$$

Definition 6.1 *A t-norm T on $[0,1]^2$ is called cancellative if for any $x \in [0,1]^2 \setminus Z =]0,1]^2$ it holds that*

$$(\forall (y,z) \in [0,1]^2 \times [0,1]^2)(T(x,y) = T(x,z) \Rightarrow y = z).$$

This definition implies that the partial mappings $T(x,\cdot)$, $x \notin Z$, of a cancellative t-norm T are strictly increasing.

Proposition 6.1 *Consider two t-norms T_1 and T_2 on $[0,1]$, then the direct product $T_1 \times T_2$ is cancellative if and only if T_1 and T_2 are cancellative.*

7 TRANSFORMATIONS OF T-NORMS

One easily verifies that any order-preserving permutation (automorphism) φ of $[0,1]^2$, i.e. a permutation of $[0,1]^2$ that satisfies

$$(\forall (x,y) \in [0,1]^2 \times [0,1]^2)(x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)),$$

transforms any t-norm T on $[0,1]^2$ into a t-norm T_φ on $[0,1]^2$ defined by

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y))).$$

Remarkably, automorphisms of the unit square are built up from automorphisms of the real unit interval, as follows: a $[0,1]^2 \to [0,1]^2$ mapping φ is an automorphism of $[0,1]^2$ if and only if there exist two automorphisms φ_1 and φ_2 of $[0,1]$ such that

$$(\forall (x,y) \in [0,1]^2)(\varphi(x,y) = (\varphi_1(x),\varphi_2(y)))$$

or

$$(\forall (x,y) \in [0,1]^2)(\varphi(x,y) = (\varphi_1(y),\varphi_2(x))).$$

One then easily verifies that for any automorphism φ of $[0,1]^2$ the transformation T_φ of the direct product $T = T_1 \times T_2$ of two t-norms T_1 and T_2 on $[0,1]$ is again a direct product of two t-norms on $[0,1]$.

8 CONTINUITY ISSUES

We have already recalled that continuous t-norms on $[0,1]$ are completely characterized as ordinal sums of continuous Archimedean t-norms. It is important to note that the notion of an ordinal sum is based on the total order of $[0,1]$ [2]. Together with the fact that the definition of the Archimedean property of t-norms on a bounded poset is not unambiguous (see Section 5), a characterization of continuous (w.r.t. the usual Borel topology) t-norms on $[0,1]^2$ similar to that of continuous t-norms on $[0,1]$ is not to be expected.

Automorphisms of the unit square do not only preserve the direct product structure of t-norms, but clearly also preserve continuity. Related to this, we have formulated the following open problem [4]: *do there exist continuous t-norms on $[0,1]^2$ that are not a direct product of continuous t-norms on $[0,1]$?* To answer this question negatively, one might first try to show that any continuous t-norm on $[0,1]^2$ has $(0,1)$ and $(1,0)$ as idempotent elements.

It is interesting to note that the continuity of a t-norm T on $[0,1]$ is equivalent with stating that

$$(\forall (x,y) \in [0,1]^2)(y \leq x \Rightarrow (\exists z \in [0,1])(T(x,z) = y)),$$

which is nothing else but a reformulation of the intermediate value theorem. This property is also called the *divisibility* of the t-norm T or the *maximal surjectivity* of T [3]. In the case of a finite chain, divisible t-norms are nothing else but the smooth t-norms of Godo and Sierra [7], as shown by Mayor and Torrens [12].

Let us call a t-norm T on $[0,1]^2$ *divisible* if for any x and y in $[0,1]^2$ the following implication holds:

$$y \leq x \Rightarrow (\exists z \in [0,1]^2)(T(x,z) = y).$$

Obviously, the direct product of two strict t-norms on $[0,1]$ is a divisible and cancellative t-norm on $[0,1]^2$. The converse, however, is not clear.

We conclude by formulating the following open problem: *do there exist divisible t-norms on* $[0,1]^2$ *that are not a direct product of continuous (i.e. divisible) t-norms on* $[0,1]$?

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