GENERATED AGGREGATION OPERATORS

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Summary

Aggregation operators generated by means of additive generators are discussed. Depending on the properties of additive generators, some classes of generated aggregation operators are derived. Several examples are included.

Keywords

Additive generator, aggregation operator, quasi-arithmetic mean, triangular norm

1. INTRODUCTION.

Two basic aggregation methods of reals are idempotent ones (min, max, arithmetic mean) and Archimedean ones (sum, bounded sum). Under some additional requirements, these methods can be shown to be the only possible up to isomorphic transformation and composition of operators (e. g. ordinal sum), see for example I-semigroups of Mostert and Shields [15], compare also [9], or quasi-arithmetic means [3]. Because of simplicity, we restrict our considerations to the unit interval inputs and outputs. However, a general real valued scale case is then a matter of rescaling only.

Definition 1. An operator

$$A: \bigcup_{n\in N}[0,1]^n \to [0,1]$$

is called an aggregation operator if it is non-decreasing, it extends the identity on [0, 1], and 0 and 1 are idempotent elements of A, i. e.,

i) $(x_1,\ldots, x_n) \le (y_1, \ldots, y_n) \Rightarrow A(x_1,\ldots, x_n) \le A(y_1, \ldots, y_n)$

ii) $A(x) = x$

and

iii) $A(0, \ldots, 0) = 0$, $A(1, \ldots, 1) = 1$.

Though a general aggregation operator A is, in fact, a collection of n-ary operators, $n \in N$, we will discuss in this contribution binary aggregation operators only (and we will call them also aggregation operators). In the case of associative operators, they fully determin the remaining n-ary operators. In general, the properties of n-ary aggregation operators, $n > 2$, are similar to those for $n = 2$. A large class of aggregation operators can be expressed by means of one-place functions which are called additive generators.

Definition 2. Let $f, g: [0, 1] \to [-\infty, \infty]$ be two continuous non-decreasing functions and let

$h: (\text{Ran } f + \text{Ran } g) \to [0, 1]$ be another continuous non-decreasing surjective function. Then the binary operator

$$A: [0, 1]^2 \to [0, 1],\ A(x, y) = h(f(x) + g(y)),$$

is an aggregation operator. The triple (f, g, h) is called a generating triple of A.

Note that A generated by a generating triple (f, g, h) is really a (binary) aggregation operator. More, A is continuous up to the case when the sum $+\infty + (-\infty)$ should be applied (in which case the non-continuity in point (0, 1) or (1, 0) occur). Though several aggregation operators can be described by means of generating triples (weighted quasi-arithmetic means, generated t-norms, t-conorms, uni-norms), the operator min, max, and related operators can never be obtained in such a way [1]. Further, if necessary, the convention fixing the value $+\infty + (-\infty)$ (always either $+\infty$ or $-\infty$) should be given.

2. SYMMETRY.

The symmetry of an operator A generated by a generating triple (f, g, h) is obviously ensured if

$$f(x) + g(y) = f(y) + g(x) \text{ for all } x, y \in [0, 1],$$

i.e., when $g = f + c$, $c = g(0) - f(0)$. However, this condition is not necessary one.

Example 1.

Define

$$f(x) = x,\ g(x) = \begin{cases} \dfrac{10x^2}{9} & \text{if } x \in [0, 0.9], \\ x & \text{if } x \in]0.9, 1] \end{cases}$$

and h(x) = max (0, 10x – 19) for $x \in [0, 2]$. Then (f, g, h) is a generating triple generating a symmetric aggregation operator A,

$$A(x, y) = \max(0, 10x + 10y - 19).$$

However, it can be shown that if a generated aggregation operator A is symmetric then it can be generated by some generated triple (f, f, h).

3. NEUTRAL ELEMENT

An aggregation operator A possesses a left unit element $e \in [0, 1]$ if A(e, x) = x for all $x \in [0, 1]$. If A is generated by a generating triple (f, g, h), then $e \in [0, 1]$ is a left unit element of A if and only if the additive generator g is strictly monotone and for all $x \in [f(e) + g(0), f(e) + g(1)]$, $h(x) = g^{-1}(x - f(e))$, i.e.,

$$h(x) = \begin{cases} 0 & \text{if } x \le f(e) + g(0) \\ 1 & \text{if } x \ge f(e) + g(1). \\ g^{-1}(x - f(e)) & \text{otherwise} \end{cases}$$

A similar claim holds in the case of a right unit element. Consequently, we get the following result.

Theorem 1. An element $e \in [0, 1]$ is a unit element of an aggregation operator A generated by a triple (f, g, h) if and only if f is strictly monotone, g = f + c (c = g(0) – f(0)), g(e) is finite and

$$h(x) = f^{-1}(\min(f(1), \max(f(0), x - g(e)))),$$

$x \in [f(0) + g(0), f(1) + g(1)]$, i.e., h is the pseudo-inverse of the mapping f + g(e), see [8].

Note that the existence of a neutral element e of a generated aggregation operator A ensures the symmetry of A. More, if e = 0 then A is t-conorm (i.e., A is associative), and if e = 1, A is a t-norm. Another associative generated aggregation operator A is a uninorm [4, 6] generated by a generating triple (f, f, f^{-1}) where f: $[0, 1] \to [-\infty, \infty]$ is a bijection (hence strictly increasing) and then $e = f^{-1}(0)$.

Example 2 (see also [14]).

Let f(x) = g(x) = x – e, $x \in [0, 1]$, $e \in [0, 1]$ and let h(x) = min (1, max(0, x + e)), $x \in [-2e, 2-2e]$. Then the corresponding generated aggregation operator A has neutral element e and is given by

$$A(x, y) = \min(1, \max(0, x + y - e)).$$

Note that A combines the Lukasiewicz t-norm T_L (which corresponds to the case e = 1) and t-conorm S_L (corresponding to e = 0) in a style described in [6]. However, the resulting operator is associative only if e= 0 or e = 1.

4. IDEMPOTENCY

The well-known class of idempotent generated aggregation operators is the class of weighted quasi-arithmetic means [3] generated by generating triples (af, bf, h), where f is some strictly monotone additive generator, a, b are positive constants and $h(x)=f^{-1}\left(\dfrac{x}{a+b}\right)$. Then the corresponding aggregation operator A is given by

$$A(x, y) = f^{-1}\left(\frac{a\,f(x) + b\,f(y)}{a+b}\right).$$

However, there are also other idempotent generated aggregation operators.

Theorem 2. An aggregation operator A generated by a generating triple (f, g, h) is idempotent, i.e., A(x, x) = x for all $x \in [0, 1]$ if and only if the sum f + g is strictly monotone and $(f + g)^{-1} = h$.

Example 3.

i) Let f(x) = x, $g(x) = x^2$. Then (f, g, h) generates an idempotent aggregation operator A only if

$$h(x)=\frac{1}{2}\left(\sqrt{1+4x}-1\right),\ x \in [0, 2],$$

in which case

$$A(x, y) = \frac{1}{2}\left(\sqrt{1+4x+4y^2}-1\right);$$

note that A is not symmetric.

ii) Let f(x) = max(0, 2x – 1), g(x) = min(2x, 1). Then neither f nor g is strictly monotone. However,

$$f(x) + g(x) = 2x,\ x \in [0, 1],$$

i.e., f + g is strictly monotone.

Consequently, for $h(x)=\frac{x}{2}$, $x \in [0, 2]$, the triple (f, g, h) is a generating triple which generates an idempotent aggregation operator A given by

$$A(x, y) = \max(0, x - \frac{1}{2}) + \min(y, \frac{1}{2}).$$

5. CONCLUSIONS

We have introduced a new class of aggregation operators covering several distinguished classes (e.g., continuous Archimedean t-norms and t-conorms, weighted quasi-arithmetic means, operators discussed in [2, 16], etc.). Several properties of introduced generated aggregation operators were discussed. Further investigations can be done for associative generated aggregation operators (compare approaches given in [2, 4, 6, 13], or for non-continuous generating triples (compare approaches given in [5, 16, 17]). Also limit properties can be of interest (compare [11, 12]) as well as the comparing of generated aggregation operators by means of the corresponding generating triples (compare [7, 10]).

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