SMOOTH ASSOCIATIVE OPERATIONS ON FINITE CHAINS

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Summary

An intuitive notion of smoothness on finite chains, introduced in [7], is investigated and formulated in a more useful mathematical way. By the help of this equivalent form, we completely characterize the class of smooth associative, increasing binary operations which also satisfy some weak boundary conditions on a chain. Some important subclasses of such operations are also described.

Keywords: ordinal scale, chain, associativity, t-norm, t-conorm, uninorm, nullnorm.

1 INTRODUCTION

One can observe an increasing interest in using linguistic scales (which are totally ordered finite sets or chains in other terminology) in several applications and theoretical problems related to fuzzy sets. Since certain types of associative operations (t-norms, t-conorms, uninorms, nullnorms, etc.) play a key role when the evaluation scale is the closed unit interval, it is important to study associative operations and their relatives (implications for instance) on linguistic scales. Some initial steps have already been made in this direction, see e.g. [7, 9, 3, 10].

The main aim of the present paper is to give the typical (general) form of associative, increasing and smooth binary operations on a finite chain. This study was motivated by a paper of Mayor and Torrens [9]. They characterized all smooth t-norms and t-conorms on finite chains. In some parts of their paper, they used a certain smoothness property borrowed from [7]. This condition is revised in Section 2, where we give a more practical, equivalent form of that smoothness. In Section 3 we determine the typical form of the mentioned operations. Note that such problem was solved by Marichal [8] when the operation was defined on a real interval. We close the paper with short descriptions of some important subclasses.

2 WHAT DOES SMOOTH MEAN ON A FINITE SET?

Assume that $L := \{x_0, x_1, \dots, x_n, x_{n+1}\}$ is a totally ordered finite set of $n+2$ elements which are indexed increasingly, according to the asymmetric and negatively transitive relation $\prec$: $x_0 \prec x_1 \prec \dots \prec x_n \prec x_{n+1}$. We use the notation $a := x_0$, $b := x_{n+1}$ in the sequel. For any $x_i, x_j \in L$, $x_i \preceq x_j$ let us define

$$\langle x_i, x_j \rangle := \{x_k \in L \mid x_i \preceq x_k \preceq x_j\},$$

which can be considered as the discrete "closed interval" of points in L between x_i and x_j (note that $x \preceq y$ if and only if either $x \prec y$, or $x = y$).

Mayor and Torrens [9] determined all associative, commutative, increasing binary operations $T : L \times L \to L$ that satisfy $T(b, b) = b$, and for all $x, y \in L$

$$x \preceq y \quad \iff \quad \exists\, z \in L \text{ such that } x = T(y, z). \quad (1)$$

We might call such a binary operation T a *smooth t-norm on L*, for obvious reasons: it can be seen that T satisfies all the four axioms of t-norms, and condition (1) is just equivalent with continuity of T when it is considered on $[0, 1]$.

In some parts of [9]), condition (1) has been substituted with the following one:

$$T(x_i, x_j) = x_r,\ T(x_{i-1}, x_j) = x_p,\ T(x_i, x_{j-1}) = x_q \\ \text{imply} \\ r - 1 \le p, q \le r. \quad (2)$$

This property was introduced in [7]. It is really a kind of smoothness: if we move only one step (up or down) from x_i or x_j, while the other argument is unchanged, then the associated function value can move (up or down) at most one step too.

One can easily find increasing operations $M : L \times L \to L$ satisfying associativity, commutativity, $M(b,b) = b$, for which condition (2) holds while (1) does not. Indeed, assume that $\mu \in L$. Let T be a smooth t-norm on $\langle \mu, b\rangle$, and define a binary operation T_μ on L as follows:

$$T_\mu(x,y) := \begin{cases} T(x,y) & \text{if } x,y \in \langle \mu, b\rangle, \\ \mu & \text{otherwise.} \end{cases} \tag{3}$$

Thus defined T_μ is a commutative, associative, increasing operation on L satisfying $T_\mu(b,b) = b$ and condition (2).

Hence, the class of associative, commutative operations M with $M(b,b) = b$ which satisfy condition (2) is wider than the class of smooth t-norms. The main difference is that a smooth t-norm satisfies $T(a,a) = a$ while T_μ does not: $T_\mu(a,a) = \mu$. Hence, for $x, y \in L$, $x \preceq y$ and $x \prec \mu$ there is no $z \in L$ such that $x = T_\mu(y,z)$, since $\mu \preceq T_\mu(y,z)$.

Because of this difference between conditions (1) and (2 for commutative operations, we want to obtain an equivalent, more usable formulation of (2). Since in our approach commutativity of M is not presupposed, we study (2) separately for the two arguments of the operation.

Definition 1. Let M be a binary operation on L. We say that M is *smooth in the first place* if

$$M(x_i,x_j) = x_r,\ M(x_{i-1},x_j) = x_p \quad \text{imply} \quad r-1 \preceq p \preceq r. \tag{4}$$

We call M *smooth in the second place* if

$$M(x_i,x_j) = x_r,\ M(x_i,x_{j-1}) = x_q, \quad \text{imply} \quad r-1 \preceq q \preceq r. \tag{5}$$

□

Now we establish equivalent formulations of (4) and (5) in the following theorem.

Theorem 1. *Let M be a binary operation on L. Condition (4) is equivalent to the following property:*

$$M(x_i,x_k) \preceq x_\ell \preceq M(x_j,x_k) \quad \textit{if and only if} \quad \exists\, x_m\colon x_i \preceq x_m \preceq x_j, \textit{ and } x_\ell = M(x_m,x_k), \tag{6}$$

and condition (5) has the following equivalent form:

$$M(x_k,x_i) \preceq x_\ell \preceq M(x_k,x_j) \quad \textit{if and only if} \quad \exists\, x_m\colon x_i \preceq x_m \preceq x_j, \textit{ and } x_\ell = M(x_k,x_m) \tag{7}$$

for all $x_k, x_i, x_j \in L$.

In view of this equivalence, the notion of smoothness of a binary operation M on L means that the so-called *intermediate-value theorem* (see [6], Lemma 1) holds in this finite case too. This makes it possible to borrow easy proof techniques from approaches using topological framework (see [6, 2]).

3 GENERAL FORM OF A SMOOTH ASSOCIATIVE BINARY OPERATION ON L

As it was mentioned, we want to find all binary operations $M : L \times L \to L$ on the finite totally ordered set $L := \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ with ordering $a := x_0 \prec x_1 \prec \ldots \prec x_n \prec x_{n+1} =: b$, which satisfy the following four conditions:

1. M is associative;
2. M is nondecreasing;
3. $M(a,a) = a$, $M(b,b) = b$;
4. M is smooth in the sense of Definition 1.

Denote $\lambda := M(b,a)$, $\mu := M(a,b)$. Since $\lambda, \mu \in L$, there are two cases: either $\lambda \preceq \mu$ or $\mu \preceq \lambda$. We consider only the first case here, the second one can be studied in a similar way. Indeed, if we have an M satisfying the previous conditions, but $\mu \preceq \lambda$, then we can apply the following results for the binary operation $M'(x,y) := M(y,x)$ $(x,y \in L)$.

First of all, some necessary conditions are formulated as a sequence of lemmas. Most of them are almost obviously true. We try to use only those properties of M that we really need in each lemma.

Lemma 1. *Suppose $M : L \times L \to L$ is associative with $M(a,a) = a$, $M(b,b) = b$. Let $\lambda := M(b,a)$, $\mu := M(a,b)$. Then we have*

$$M(\lambda,a) = M(b,\lambda) = \lambda, \tag{8}$$
$$M(a,\mu) = M(\mu,b) = \mu. \tag{9}$$

The next lemma states that associativity and increasingness completely determine values of M, which are constant, in two rectangles around the corners (a,b) and (b,a). In particular, we can see that $M(\lambda,\lambda) = \lambda$ and $M(\mu,\mu) = \mu$.

Lemma 2. *In addition to conditions in Lemma 1, suppose M is increasing and $\lambda \preceq \mu$. Then we have*

$$M(x,y) = \lambda \quad \textit{for } x \in \langle \lambda, b\rangle,\ y \in \langle a, \lambda\rangle, \tag{10}$$
$$M(x,y) = \mu \quad \textit{for } x \in \langle a, \mu\rangle,\ y \in \langle \mu, b\rangle, \tag{11}$$

By the next statement we can learn in particular that taking the restriction of M to the square $\langle a,\lambda\rangle \times \langle a,\lambda\rangle$, a is the identity of this restriction (like 0 for a t-conorm), while restricting M to $\langle \mu,b\rangle \times \langle \mu,b\rangle$, b is the identity of this second restriction (like 1 for a t-norm).

Lemma 3. *Suppose $M : L \times L \to L$ is associative, increasing and smooth with $M(a,a) = a$, $M(b,b) = b$. Let $\lambda := M(b,a)$, $\mu := M(a,b)$. Then we have*

$$M(a,y) = y \text{ for all } y \in \langle a,\mu\rangle, \quad (12)$$
$$M(b,y) = y \text{ for all } y \in \langle \lambda,b\rangle. \quad (13)$$

We emphasize that symmetry (or commutativity) is not prescribed for M. Nevertheless, on the squares $\langle a,\lambda\rangle \times \langle a,\lambda\rangle$ and $\langle \mu,b\rangle \times \langle \mu,b\rangle$ M turns out to be commutative, as a consequence of the following lemma.

Lemma 4. *Suppose $M : L \times L \to L$ is associative, increasing, smooth, and $M(b,x) = M(x,b) = x$ for all $x \in L$. Then M is commutative on L.*

Thus, applying this lemma to $\langle \mu,b\rangle \times \langle \mu,b\rangle$, and the same for the dual operation on $\langle a,\lambda\rangle \times \langle a,\lambda\rangle$, we can conclude that the previous one is a smooth t-norm, and the second one is a smooth t-conorm. Therefore, the only missing part of $L \times L$ is described now.

Lemma 5. *Suppose $M : L \times L \to L$ is associative, increasing and smooth with $M(a,a) = a$, $M(b,b) = b$. Let $\lambda := M(b,a)$, $\mu := M(a,b)$. Then we have*

$$M(x,y) = y \quad \text{for all } x \in L,\ y \in \langle \lambda,\mu\rangle. \quad (14)$$

We summarize the previous lemmas in the "necessity" part of the following theorem. Before that we introduce S_λ as follows:

$$S_\lambda(x,y) := \begin{cases} S(x,y) & \text{if } x,y \in \langle a,\lambda\rangle, \\ \lambda & \text{otherwise} \end{cases} . \quad (15)$$

Theorem 2. *Consider a function $M : L \times L \to L$ with properties $M(a,a) = a$, $M(b,b) = b$. Let $\lambda := M(b,a)$, $\mu := M(a,b)$, and suppose $\lambda \preceq \mu$. If M is associative, increasing and smooth then M is of the following form $(x,y \in L)$:*

$$M(x,y) = (S_\lambda(x,y) \wedge y) \vee (T_\mu(x,y) \wedge y) \vee (x \wedge y), \quad (16)$$

where functions T_μ, S_λ are defined by (3) and (15), respectively.

The typical form of the operator M is given in Figure 1 when $\lambda \preceq \mu$, and in Figure 2 for the case of $\mu \preceq \lambda$, for sake of simplicity, on the unit square.

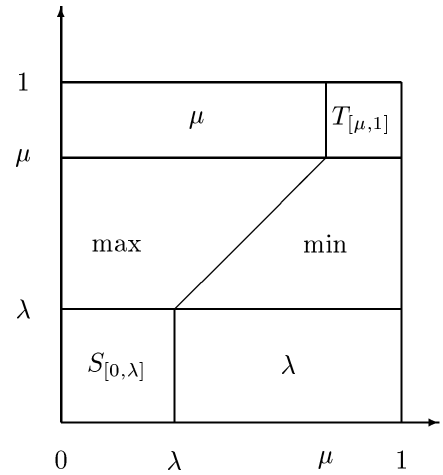

Figure 1: M in case of $\lambda \preceq \mu$

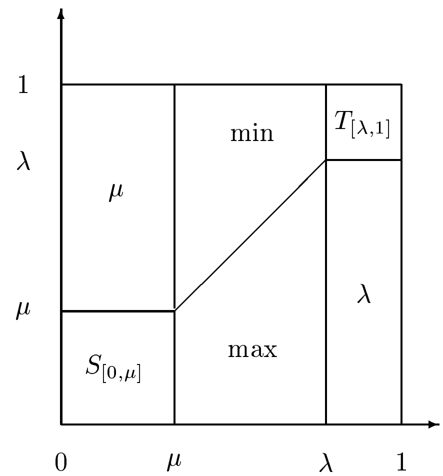

Figure 2: M in case of $\mu \preceq \lambda$

4 PARTICULAR CLASSES

Now we investigate some particular classes of the general case characterized in the previous section. We can recognize subclasses corresponding to well-known families of associative binary operations on $[0,1]$. Denote $\mathcal{C}_{\langle a,b\rangle}$ the class of all associative, increasing, smooth binary operations on $L = \langle a,b\rangle$ (in other words: the class of functions having form (16)).

4.1 COMMUTATIVE OPERATIONS

By Theorem 2, if $M \in \mathcal{C}\langle a,b\rangle$ is commutative then $\lambda = M(b,a) = M(a,b) = \mu$. On the other hand, it is obvious from Eq. (16) that $\lambda = \mu$ is sufficient at the same time to ensure commutativity of $M \in \mathcal{C}\langle a,b\rangle$. Therefore, we obtain that the commutative subclass of

$\mathcal{C}\langle a,b\rangle$ on L is exactly the class of *nullnorms* defined explicitely in [1]. Remark that, in other terminology, this subclass corresponds to associative $T-S$ aggregation functions introduced and investigated in [4]. Note that in [10] so-called *t-operators* have been studied and completely characterized. These operators are exactly the commutative members of $\mathcal{C}\langle a,b\rangle$.

4.2 IDEMPOTENT OPERATIONS

Let $M \in \mathcal{C}\langle a,b\rangle$ be idempotent: $M(x,x) = x$ for all $x \in \langle a,b\rangle$. Then, both $T_{\langle a,\mu\rangle}$ and $S_{\langle \lambda,b\rangle}$ are idempotent, whence $T_{\langle a,\mu\rangle} = \min_{\langle a,\mu\rangle}$ and $S_{\langle \lambda,b\rangle} = \max_{\langle \lambda,b\rangle}$. Consequently, M can be written in the following form:

$$M(x,y) = (\lambda \wedge x) \vee (\mu \wedge y) \vee (x \wedge y) \quad (x,y \in L). \tag{17}$$

This operation was introduced and characterized in [2] as the noncommutative extension of the median (see [6]) in connected order topological spaces. Fortunately, the characterization of (17) is still valid in the present finite ordinal framework.

4.3 OPERATIONS HAVING ABSORBING ELEMENT

Suppose $M \in \mathcal{C}\langle a,b\rangle$ and M has an *absorbing element* $z \in L$: $M(x,z) = M(z,x) = z$ for all $x \in L$. Then, by Lemma 2, we have that $z = \lambda = \mu$, whence M is commutative. Therefore, the subclass $M \in \mathcal{C}\langle a,b\rangle$ with absorbing element is the same as the subclass of commutative operations.

4.4 OPERATIONS HAVING NEUTRAL ELEMENT

Suppose $M \in \mathcal{C}\langle a,b\rangle$ and M has a *neutral element* $e \in L$: $M(x,e) = M(e,x) = x$ for all $x \in L$ (this case corresponds to *uninorms* on $[0,1]$, see [5]). Then, again by Lemma 2, e cannot be strictly between a and b. If $e = a$ then we obtain smooth t-conorms, and if $e = b$ we get smooth t-norms. Therefore, these are the two subclasses of $\mathcal{C}\langle a,b\rangle$ having neutral element. Note that this fact is also proved in [10]. Moreover, similarly to the case of $[0,1]$, pseudo-smooth uninorms on a finite chain are introduced and characterized there.

5 CONCLUSION

In this paper we have studied a class of associative, increasing and smooth binary operations defined on a finite chain. We have shown the general form of such operations, and investigated some particular subclasses too.

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