Characterization of commonotone aggregation operators

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Summary

We consider the aggregation operators which are $\oplus$-additive for commonotone functions. The main result is that any operator is expressed by a kind of general fuzzy integral which uses a family of fuzzy measures connected by a $\oplus$-additive Cauchy's equations. In particular, the family of fuzzy measures can be obtained by a $\oplus$-fitting pseudo-multiplication. In this case, the aggregation operator is exactely expressed by a general fuzzy integral [1].

Keywords: Aggregation, Commonotone, Fuzzy integral, Cauchy's equation.

1 INTRODUCTION

Let Ω be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of Ω and $\mathcal{F}$ the family of all $\mathcal{A}$-measurable functions $f : \Omega \to [0, F]$, with $0 < F \leq +\infty$. A chain $\mathcal{M}_f \supset \mathcal{A}$ is associated to any $f \in \mathcal{F}$: it is constituted by the sets

$$C_f(x) =: \{\omega \in \Omega \mid f(\omega) > x\} \qquad x \in [0, F]\,.$$

We suppose that the interval $[0, F]$ has a structure of I-semigroup defined by means of a *pseudo-addition* $\oplus$.

The binary operation $\oplus : [\,0, F\,]^2 \to [\,0, F\,]$ is called pseudo-addition if it is commutative, associative, monotone non decreasing, continuous, and if 0 is its neutral element.

The structure of the pseudo-additions is known; see [7] or [1] for the general form of the operation $\oplus$. We shall use, as well, a *pseudo-difference* defined for $a < b$ by

$$b \ominus a = \inf\{x \in [0, F] \mid a \oplus x = b\}\,.$$

The family $\mathcal{F}$ of the measurable functions is generated by the set $\mathcal{B}$ of the *basic functions*. For any $a \in [0, F]$ and $A \in \mathcal{A}$, a basic function $b(a, A)$ is defined by:

$$b(a, A)(\omega) = \begin{cases} a & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases} \tag{1}$$

An element s of $\mathcal{F}$ is called *simple function* if its range is finite. The simple functions have several representations by means of basic functions. The well-known classical one is:

$$s = \bigvee_{i=1}^{n} b(a_i, A_i) \tag{2}$$

with $A_i \cap A_j = \phi$ for $i \neq j$ and $0 < a_1 < a_2 < \ldots < a_n$. Moreover, any simple function admits many $\oplus$-*step representations*:

$$s = \bigoplus_{i=1}^{m} b(c_i, C_i) \tag{3}$$

with $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_m$. The standard $\oplus$-step representation is obtained from (2) assuming $m = n$ and:

$$c_1 = a_1\,, \quad c_2 = a_2 \ominus a_1, \ldots, c_n = a_n \ominus a_{n-1}$$

$$C_i = \bigcup_{j=1}^{n} A_j = \{\omega \in \Omega \mid s(\omega) > a_i\}\,.$$

The standard $\oplus$-step representation is minimal for the number of steps and for the heigt of the single steps.

Let $\mathcal{S}$ be the set of all simple functions, we put

$$\mathcal{S}_f =: \{s \in \mathcal{S} \mid s \leq f\}.$$

For every $f \in \mathcal{F}$ it is:

$$f(\omega) = \sup\{s(\omega) \mid s \in \mathcal{S}_f\} \tag{4}$$

because every $f \in \mathcal{F}$ is the limit of a increasing sequence of simple functions ([5] Ch. IV)

It is useful to recall the definition of commonotony for functions [4]. Any function $f : \Omega \to \mathbb{R}$ introduces on Ω a semi-order relation:

$$\omega_1 <_f \omega_2 \Longleftrightarrow f(\omega_1) < f(\omega_2)\,.$$

The functions f and g are called *commonotone* ($f \sim g$) if the two corresponding semi-order are non-contradictory, i.e., there exists no pair ω_1, ω_2 in Ω such that $\omega_1 <_f \omega_2$ and $\omega_2 <_g \omega_1$.

We remark that the $\oplus$-step representation of a simple function is built adding commonotone functions. Therefore, the $\oplus$-step representation is called $\oplus$-*commonotone additive representation* too.

2 AGGREGATION OPERATORS

We shall consider the commonotone aggregation operators, i.e., the functionals $L : \mathcal{F} \to [0, F]$ satisfying the following properties:

(L1) $\quad f(\omega) \le f'(\omega) \Longrightarrow L(f) \le L(f')$

(monotonicity)

(L2) $\quad f(\omega) = c \quad \forall\, \omega \in \Omega \Longrightarrow L(f) = c$

(idempotence)

(L3) $\quad f_n \uparrow f \Longrightarrow L(f_n) \uparrow L(f)$

(continuity from below)

(L4) $\quad f_1 \sim f_2 \Longrightarrow L(f_1 \oplus f_2) = L(f_1) \oplus L(f_2)$

(commonotone $\oplus$-additivity).

The properties (L1), (L2), (L3) seem us to be natural for the aggregation operators ([6], [8], [9]) because they are evaluation models or averaging operations. The commonotone $\oplus$-additivity fits the operators to the structure of the semigroup acting on the range of values that shall be aggregated.

For these operators we shall recognize that they are completely determined by their values on the set $\mathcal{B}$, i.e., by the function

$$\ell(a, A) = L[b(a, A)]\,. \qquad (5)$$

2.1. Lemma. *Given a commonotone aggregation operator L, for any $a \in]0, F]$ the map $\mu_a : \mathcal{A} \to [0, F]$ defined by*

$$\mu_a(A) =: \ell(a, A) \qquad (6)$$

is a fuzzy measure continuous from below, with $\mu_a(\Omega) = a$.

2.2. Lemma. For any $\oplus$-step representation (3) of a given simple function s it holds

$$L(s) = \bigoplus_{i=1}^{m} \ell(c_i, C_i)\,. \qquad (7)$$

2.3. Lemma. *For every $f \in \mathcal{F}$ it is:*

$$L(f) = \sup\{L(s) \mid s \in \mathcal{S}_f\}\,. \qquad (8)$$

The lemmas above are direct consequences of [(L1)-(L4)]; they show that the family of fuzzy measures $\{\mu_a \mid a \in]0, F]\}$ is associated in biunivocal way to the corresponding operator.

2.4. Proposition. *The family $\{\mu_a\}$, associated to a commonotone aggregation operator, satisfies the following properties:*

(F1) $\quad \mu_a : \mathcal{A} \to [0, F]$ *is a fuzzy measure*

continuous from below $\forall\, a \in]0, F]$.

(F2) $\quad \mu_a(\Omega) = a \quad \forall\, a \in]0, F]$

(F3) $\quad a < b \Longrightarrow \mu_a \le \mu_b$

(F4) $\quad \mu_{a \oplus b} = \mu_a \oplus \mu_b \quad \forall\, a, b \in]0, F]$.

The last property is a link between the measures of every element $A \in \mathcal{A}$. It is expressed by a $\oplus$-additive Cauchy's generalised equation [3].

3 CONSTRUCTION OF COMMONOTONE AGGREGATION OPERATORS

The properties of $\{\mu_a\}$, seen in proposition 2.4, are characteristic of the families of fuzzy measures which correspond to the commonotone $\oplus$-additive operators. This is shown by the following

3.1. Theorem. *One commonotone aggregation operator enjoing* [(L1)-(L4)] *is associated to every family of fuzzy measures satisfying* [(F1)-(F4)].

The proof of this theorem is laborious but it uses of a quite natural course inspired by the lemmas 2.2 and 2.3. Now we show the line of the demonstration enouncing some lemmas. The proofs of these lemmas are similar to those used in [1] for the construction of the integral with respect to a general fuzzy measure.

Let $\{\mu_a\}$ *be a family of fuzzy measures satisfying* [(F1)-(F4)].

3.2. Lemma *For every simple function s the espression*

$$L(s) = \bigoplus_{i=1}^{m} \mu_{c_i}(C_i) \tag{9}$$

has the same value for every its $\oplus$*-step representation.*

Therefore the expression (9) can be taken as definition of a functional on the set $\mathcal{S}$.

3.3. Lemma. *The functional* (9) *is non decreasing, continuous from below and* $\oplus$*-additive.*

3.4. Lemma. *For every* $f \in \mathcal{F}$ *the expression* (8) *with* $L(s)$ *given by* (9) *defines an aggregation operator satisfying the properties* [(L1)-(L4)].

We end the proof of the theorem 3.1 remarking that the construction given by the lemmas 3.2 and 3.4 is the only possible. Every commonotone aggregation operator is a kind of fuzzy general integral built from a suitable family of fuzzy measures.

4 AGGREGATION OPERATORS AS GENERAL INTEGRALS

We can construct a family $\{\mu_a\}$ from a given fuzzy measure m by means of a pseudo-multiplication.

Let $\oplus$ be a given pseudo-addition on $[0, F]$. A binary operation $\odot$: $[0, F] \times [0, M] \to [0, F]$ is called a $\oplus$-fitting *pseudo-multiplication* [1] if the following properties are satisfied:

(M1) $$a \leq a', \nu \leq \nu' \Longrightarrow a \odot \nu \leq a' \odot \nu'$$

(monotonicity)

(M2) $$a \odot 0 = 0 \odot \nu = 0$$

(zero element)

(M3) $$(\sup a_n) \odot (\sup \nu_m) = \sup(a_n \odot \nu_m)$$

(left continuity)

(M4) $$(a \oplus b) \odot \nu = (a \odot \nu) \oplus (b \odot \nu)$$

(left distributivity).

Let $m : \mathcal{A} \to [0, M]$ $(0 < M \leq +\infty)$ be a fuzzy measure continuous from below, we assume:

$$\mu_a(A) = a \odot m(A) \tag{10}$$

4.1. Proposition. *If* $m(\Omega)$ *is right unitary element for the pseudo-multiplication, the family* (10) *of fuzzy measures satisfies the properties* [(F1)-(F4)].

The corresponding operators is the general integral [2], [1],

$$L(f) = \int^{\oplus} f \odot dm\,. \tag{11}$$

We have the integral form for the aggregation operator if one of the fuzzy measures in the family $\{\mu_a\}$ is discriminant in camparison with the others.

4.2. Definition *The fuzzy measure* μ *is* discriminant *in the family* $\{\mu_a\}$ *if*

$$\mu(A) = \mu(A') \Longrightarrow \mu_c(A) = \mu_c(A') \quad \forall\, c \in [0, F].$$

4.3. Proposition. *If there exists a value* $u \in]0, F]$ *such that the corresponding fuzzy measure* μ_u *is discriminant in the family* $\{\mu_a\}$*, then there exists a* $\oplus$*-fitting pseudo-multiplication so that*

$$\mu_c(A) = c \odot \mu_u(A)\,. \tag{12}$$

and the expression (11) *holds, with* $m = \mu_u$.

Proof. From the definition 4.2, $\mu_c(A)$ is a function of c and A, which depends, realy, from c and $\mu_u(A)$:

$$\mu_c(A) = \varphi[c, \mu_u(A)].$$

The operation $\odot$ is defined by putting

$$\begin{cases} 0 \odot m = 0 \\ c \odot m = \varphi[c, m] \quad \forall\, c \neq 0\,. \end{cases}$$

The pseudo-multiplication satisfies the properties [(M1)-(M4)] and it has u as unit left element, because

$$u \odot \mu_u(A) = \varphi[u, \mu_u(A)] = \mu_u(A)\,.$$

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