(S, U)-integral

Erich Peter Klement

Institute of Mathematics, Johannes Kepler University, 4040 Linz, Austria

e-mail: klement@flll.uni-linz.ac.at

Radko Mesiar

Slovak Technical University, 813 68 Bratislava, Slovakia

and ÚTIA AV ČR, *P.O.Box 18, 182 08 Praha, Czech republic*

e-mail: mesiar@cvt.stuba.sk

Endre Pap

Institute of Mathematics, University of Novi Sad, 21000 Novi Sad, Yugoslavia

e-mail: pap@unsim.ns.ac.yu , pape@eunet.yu

Key words: t-conorm, uninorm, S-measure, (S, U)-integral.

1 Introduction

It is introduced an integral based on t-conorm S, a uninorm U and an S-measure (decomposable measure) with the aim to cover almost all known integrals constructed by pseudo-operations.

A triangular conorm (t-conorm) S is a commutative, associative, monotone binary operation on the unit interval $[0, 1]$ with $S(x, 0) = x$ for all $x \in [0, 1]$. The four basic t-conorms are

$$S_{\mathbf{M}}(x, y) = \max(x, y), \quad S_{\mathbf{P}}(x, y) = x+y-xy,$$

$$S_{\mathbf{L}}(x, y) = \min(x + y, 1),$$

$$S_{\mathbf{D}}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

2 S-measures

Let X be a fixed non-empty set.

Definition 1 *Let S be a t-conorm and let $\mathcal{A}$ be a σ-algebra of subsets of X. A mapping $m : \mathcal{A} \to [0, 1]$ is called an S-measure if it is continuous from below, $m(\emptyset) = 0$ and if m is S-decomposable, i.e., for all $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ we have*

$$m(A \cup B) = S(m(A), m(B)).$$

Remark 2 (i) If S is a left continuous t-conorm, then a set function $m : \mathcal{A} \to [0, 1]$ satisfying $m(\emptyset) = 0$ is an S-measure if and only if for each sequence $(A_n)_{n\in\mathbf{N}}$ of pairwise disjoint elements of $\mathcal{A}$ we have

$$m\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mathop{\mathrm{S}}_{n=1}^{\infty} m(A_n).$$

(ii) If the set X is finite or countably infinite and S is a left continuous t-conorm, then each S-measure $m : \mathcal{P}(X) \to [0, 1]$ is uniquely determined by the values $m(\{x\})$ with $x \in X$.

(iii) A set function $m : \mathcal{A} \to [0, 1]$ is $S_{\mathbf{M}}$-decomposable if and only if for all $A, B \in \mathcal{A}$ we have

$$m(A \cup B) = S_{\mathbf{M}}(m(A), m(B)).$$

Example 3 (i) Obviously, each measure $m : \mathcal{A} \to [0, \infty]$ with Range$(m) \subseteq [0, 1]$ is an $S_{\mathbf{L}}$-measure.

(ii) The function $m : \mathcal{B} \to [0, 1]$ defined by $m(A) = \min(\lambda(A), 1)$, where $\mathcal{B}$ is the σ-algebra of all Borel subsets of $\mathbb{R}$ and $\lambda : \mathcal{B} \to [0, \infty]$ is the Lebesgue measure, is an $S_{\mathbf{L}}$-measure, but not a measure.

(iii) Fixing a number $K \in \mathbb{N}$ and defining the set function $m : \mathcal{P}(\mathbb{N}) \to [0, 1]$ by

$$m(A) = \min\Big(\frac{\text{card}(A)}{K}, 1\Big),$$

where card(A) denotes the cardinality of the set A, it is easily seen that m is always an $S_{\mathbf{L}}$-measure, and an $S_{\mathbf{D}}$-measure in the case $K = 2$, but not a measure.

(iv) The function $m : \mathcal{B} \to [0, 1]$ defined by

$$m(A) = 1 - e^{-\lambda(A)}$$

is an $S_{\mathbf{P}}$-measure, for λ as in (ii).

(v) For an arbitrary function $f : X \to [0, 1]$, the set function $m_f : \mathcal{P}(X) \to [0, 1]$ defined by $m_f(A) = \sup\{f(x) \mid x \in A\}$ is an $S_{\mathbf{M}}$-measure.

(vi) The set function $m : \mathcal{P}([0, 1]) \to [0, 1]$ defined by

$$m(A) = \begin{cases} \sup A & \text{if } A \text{ is finite or countably infinite,} \\ 1 & \text{otherwise,} \end{cases}$$

is an $S_{\mathbf{M}}$-measure.

(vii) Let $P : \mathcal{A} \to [0,1]$ be a probability measure. Then for each $\alpha \in]-1, \infty[\setminus \{0\}$ the set function $m_\alpha : \mathcal{A} \to [0,1]$ defined by

$$m_\alpha(A) = \frac{1}{\alpha}\left((1+\alpha)^{P(A)} - 1\right)$$

is an $S_\alpha^{\mathbf{SW}}$-measure for the Sugeno-Weber t-conorm $S_\alpha^{\mathbf{SW}}(x,y) = \min(x + y + \alpha xy, 1)$. . Observe that we have $m_\lambda = (s_\alpha^{\mathbf{SW}})^{-1} \circ P$, where $s_\alpha^{\mathbf{SW}}$ is the unique additive generator of $S_\alpha^{\mathbf{SW}}$ which satisfies $s_\alpha^{\mathbf{SW}}(1) = 1$ (compare [16]). We remark that m_λ is subadditive if and only if $\lambda \in]-1, 0[$, and superadditive if and only if $\lambda \in]0, \infty[$.

(viii) If $m : \mathcal{A} \to [0,1]$ is an S-measure and $\psi : [0,1] \to [0,1]$ an increasing bijection, then $\psi^{-1} \circ m : \mathcal{A} \to [0,1]$ is an S_ψ-measure, where S_ψ is defined by

$$S_\psi(x,y) = \psi^{-1}\left(S(\psi(x), \psi(y))\right).$$

The S-measures presented in Example 3(i)–(iv),(vii) are special cases of (σ-)*S-decomposable measures* [1, 18], some of which can be constructed as follows:

Proposition 4 *Let S be a continuous Archimedean t-conorm, $s : [0,1] \to [0,\infty]$ an additive generator of S, $\mathcal{A}$ a σ-algebra of subsets of X and $m : \mathcal{A} \to [0,\infty]$ a measure. Then the function $s^{(-1)} \circ m : \mathcal{A} \to [0,1]$ is an S-measure (here $s^{(-1)}$ is the pseudo-inverse of s).*

Definition 5 *Let S be a t-conorm, $\mathcal{A}$ a σ-algebra of subsets of X and $m : \mathcal{A} \to [0,1]$ an S-measure. A set $A \in \mathcal{A}$ is called S-m-faithful if for each $B \in \mathcal{A}$ with $B \subseteq A$ such that, whenever $(u,v) \in [0, m(B)[\times [0, m(A \setminus B)[$, we have $S(u,v) < 1$. A partition $\mathcal{C} = \{C_k \mid C_k \in \mathcal{A}, k \in K\}$ of X, where K is finite or countably infinite , is called an S-m-partition if, for each $k \in K$, the set C_k is S-m-faithful. The S-measure is said to be S-faithful if there exists an S-m-faithful partition $\mathcal{C}$ of X.*

Remark 6 Let S be a t-conorm, $\mathcal{A}$ a σ-algebra of subsets of X and $m : \mathcal{A} \to [0,1]$ an S-measure.
(i) If $A \in \mathcal{A}$ is S-m-faithful then each $B \in \mathcal{A}$ with $B \subseteq A$ is S-m-faithful.
(ii) Each $A \in \mathcal{A}$ with $m(A) < 1$ is S-m-faithful.
(iii) If we have $S(x,y) = 1$ if and only if $\max(x,y) = 1$ then X is S-m-faithful. Consequently, each $S_{\mathbf{M}}$-measure is $S_{\mathbf{M}}$-faithful.
(iv) If S is a nilpotent t-conorm, then the S-faithfulness of an S-measure is a weaker property than the m-achievability of X, see [18].

An interesting relationship between (σ-additive) measures and S-measures with respect to some continuous Archimedean t-conorm S is given as follows:

Proposition 7 *Let S be a continuous Archimedean t-conorm, $s : [0,1] \to [0,\infty]$ an additive generator of S, $\mathcal{A}$ a σ-algebra of subsets of X and $m : \mathcal{A} \to [0,1]$ an S-measure. Then $s \circ m : \mathcal{A} \to [0,\infty]$ is a measure if and only if X is S-m-faithful.*

The assumption that X is S-m-faithful cannot be dropped because of Example 3 (ii),(iii). Observe that, in (ii), a set $A \in \mathcal{B}$ is $S_{\mathbf{L}}$-m-faithful if and only if $\lambda(A) \leq 1$, and in (iii), if and only if $\mathrm{card}(A) \leq K$.

3 (S,U)-integral

A uninorm U (respectively t-norm T) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$ and for some $0 < e < 1$ we have $U(x,e) = x$ (respectively $T(x,1) = x$) for all $x \in [0,1]$. Throughout this section, let U be a left-continuous uninorm or t-norm and S a continuous t-conorm such that U is conditionally distributive over S, i.e., they satisfy the property (CD):

$$U(x, S(y,z)) = S(U(x,y), U(x,z))$$

for all $x,y,z \in [0,1]$ such that $S(y,z) < 1$.

Denote by $\mathcal{M}$ the set of all $\mathcal{A}$-measurable functions from X to $[0,1]$. As usual, a measurable step function $\varphi : X \to [0,1]$ is a measurable function which assumes only finitely many values, and the set of all step functions will be denoted $\mathcal{S}$. If $\mathrm{Range}(\varphi) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $\alpha_i \neq \alpha_j$ whenever $i \neq j$, and if $A_i = \varphi^{-1}(\{\alpha_i\})$, then there is a canonical representation of φ given by

$$\varphi = \mathop{\mathrm{S}}_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}. \tag{1}$$

Observe that representation (1) is independent of the t-conorm S under consideration. Let $m : \mathcal{A} \to [0,1]$ be an S-faithfull S-measure in whole section.

Definition 8 *(i) The (S,U)-integral of a measurable step function $\varphi : X \to [0,1]$ (which is represented as in (1)) is defined by*

$$\int_X^{(S,U)} \varphi \, dm = \mathop{\mathrm{S}}_{k\in K} \Big(\mathop{\mathrm{S}}_{i=1}^{n} U(\alpha_i, m(A_i \cap C_k)) \Big),$$

where $\mathcal{C}$ is an S-m-partition.

(ii) The (S,U)-integral of a measurable function $f : X \to [0,1]$ is defined by

$$\int_X^{(S,U)} f \, dm = \sup \Big\{ \int_X^{(S,U)} \varphi \, dm \mid \varphi \in \mathcal{S}, \varphi \le f \Big\}.$$

(iii) The (S,U)-integral of a measurable function $f : X \to [0,1]$ on a set $A \in \mathcal{A}$ is defined by

$$\int_A^{(S,U)} f \, dm = \int_X^{(S,U)} f_A \, dm,$$

where the function f_A is equal f on the set A and otherwise it is zero.

Theorem 9 *For all $f, h \in \mathcal{M}$ we have*

$$\int_X^{(S,U)} f \, dm \le \int_X^{(S,U)} h \, dm$$

whenever $f \le h$,

$$\int_X^{(S,U)} S(f,h) \, dm = S\Big(\int_X^{(S,U)} f \, dm, \int_X^{(S,U)} h \, dm \Big),$$

whenever for all $a \in [0,1]$ we have $U(S(f,h),a) = S(U(f,a), U(h,a))$.

Theorem 10 *For each $f \in \mathcal{M}$, the function $m_f : \mathcal{A} \to [0,1]$ defined by*

$$m_f(A) = \int_A^{(S,U)} f \, dm,$$

is an S-faithful S-measure.

On the family

$$\{f \in \mathcal{M} \mid X \text{ is } S - m_f \text{ -faithful}\}$$

the (S,U)-integral is U-homogeneous.

For the (S,U)-integral some classical convergence theorems such as the monotone convergence theorem and Fatou's lemma, can be proven, as well as a Radon-Nikodým theorem for the (S,U)-integral [4] (compare also [5, 13, 17]).

For special uninorms or t-norms U and t-conorms S, we obtain some integrals which are well-known in the literature.

Example 11 (i) Let $m : \mathcal{A} \to [0,1]$ be an $S_{\mathbf{L}}$-measure which is also a (σ-additive) measure. Then the $(S_{\mathbf{L}}, T_{\mathbf{P}})$-integral of a function $f \in \mathcal{M}$ coincides with the classical Lebesgue integral of f.

(ii) For each left-continuous U, for each $S_{\mathbf{M}}$-measure $m : \mathcal{A} \to [0,1]$ and for each function $f \in \mathcal{M}$ we have that $(S_{\mathbf{M}}, U)$-integral is given by

$$\sup \big\{ U(a, m(\{x \in X \mid f(x) \ge a\})) \bigm| a \in [0,1] \big\}.$$

If m is a completely maxitive measure then we have

$$\int_X^{(S_{\mathbf{M}},U)} f \, dm = \sup\{U(f(x), m(\{x\})) \mid x \in X\}.$$

In particular, the $(S_{\mathbf{M}}, T_{\mathbf{M}})$-integral coincides with the integral introduced in [16], and the $(S_{\mathbf{M}}, T_{\mathbf{P}})$-integral coincides with the integral studied in [15].

(iii) The restriction to S-measures and measurable functions whose ranges are subsets of $[0,1]$ causes no loss of generality. Let $([a,b], \oplus, \odot)$, for $[a,b] \subseteq [-\infty, \infty]$, be a continuous semiring, see [2], with zero element a (see [3]). Consequently, by [3], there is some strictly increasing bijection $\psi : [a,b] \to [0,1]$ such that the continuous t-conorm S defined by

$$S(x,y) = \psi(\psi^{-1}(x) \oplus \psi^{-1}(y))$$

is isomorphic to $\oplus$. Define the operation (uninorm or t-norm) U on $[0,1]$ by

$$U(x,y) = \psi(\psi^{-1}(x) \odot \psi^{-1}(y))$$

and observe that U is conditionally distributive over S. Then for an $\mathcal{A}$-measurable function $f : X \to [a,b]$ and an $\oplus$-faithful $\oplus$-measure $m : \mathcal{A} \to [a,b]$ one can introduce $(\oplus, \odot)$-integral in complete analogy to our construction. Such integral was introduced in [8, 11] and can be represented by the (S,U)-integral via

$$\int_X^{(\oplus,\odot)} f \odot dm = \psi^{-1}\Big(\int_X^{(S,U)} (\psi \circ f) \, d(\psi \circ m) \Big).$$

(iv) For $[a,b] = [0, \infty]$ we obtain an integral introduced in [17], and for $([a,b], \oplus, \odot) = ([0,\infty], +, \cdot)$ we again come back to the classical Lebesgue integral.

(v) If the operations $\oplus$ and $\odot$ in the semiring $([a,b], \oplus, \odot)$ are generated by some strictly increasing bijection $g : [a, b] \to [0, \infty]$ via

$$x \oplus y = g^{-1}(g(x) + g(y)),$$
$$x \odot y = g^{-1}(g(x)g(y)),$$

the integral introduced in [11] (called g-integral in [7, 12]) has the special form

$$\int_X^{(\oplus,\odot)} f \odot \, dm = g^{-1}\Big(\int_X (g \circ f) \, d(g \circ m) \Big).$$

Under some additional conditions, also a g-derivative can be defined [12] which turns out to be useful when solving some differential equations (see [8, 13]).

(vi) If S is a continuous Archimedean t-conorm with an additive generator $s : [0, 1] \to [0, \infty]$ and if we replace U by the binary operation $*$ given by

$$a * b = s^{-1}(a \cdot s(b))$$

we can repeat the whole construction of Definition 8 for the $(S, *)$-integral, and we obtain the integral introduced in [18].

Further generalizations and modifications of the (S, U)-integral can be found in [5, 6, 8, 9, 10, 13, 14].

Acknowledgement

The third author would like to thank the Department of Mathematics of the Johannes Kepler University, Linz, Austria, for the financial support of their visits. The second author was supported by grants VEGA 2/6087/99, GACR 402/99/032 and the first author with the second author were supported also by Action Austria-Slovakia 26s34.

References

[1] D. Dubois, H. Prade, A class of fuzzy measures based on triangular norms, Internat. J. Gen. System 8 (1982), 43–61.

[2] J.S. Golan, *The theory of semirings with applications in mathematics and theoretical computer science*, Longman, 1992.

[3] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms* (to appear).

[4] E.P. Klement, R. Mesiar, E. Pap, (S, U)-integral based on S-measure (preprint).

[5] E.P. Klement, S. Weber, Generalized measures, Fuzzy Sets and Systems 40 (1991), 375–394.

[6] A. Kolesárová, A note to the $\oplus$-measure based integrals, Tatra Mt. Math. Publ. 3 (1993), 173-182.

[7] A. Marková, A note on g-derivative and g-integral, Tatra Mt. Math. Publ. 8 (1996), 71-76.

[8] V.P. Maslov, S.N. Samborskij (eds.), *Idempotent Analysis*, Advances in Soviet Mathematics 13, Amer. Math. Soc., Providence, Rhode Island, 1992.

[9] R. Mesiar, Choquet–like integrals, J. Math. Anal. Appl, 194 (1995), 477–488.

[10] T. Murofushi, M. Sugeno, Fuzzy t–conorm integrals with respect to fuzzy measures: generalization of Sugeno integral and Choquet integral, Fuzzy Sets and Systems 42 (1991), 57–71.

[11] E. Pap, Integral generated by decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20,1 (1990), 135-144.

[12] E. Pap, $g-$calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23,1 (1993), 145-156 .

[13] E.Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[14] M. Sion, *A Theory of Semigroup Valued Measures*, Springer-Verlag, 1973.

[15] N. Shilkret, Maxitive measure and integration, Indag. Math. 3 (1971) (1971), 109-116.

[16] M. Sugeno, Theory of fuzzy integrals and its applications, Ph. D. Thesis, Institute of Technology, Tokyo, 1974.

[17] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987), 197-222.

[18] S. Weber, $\perp -$decomposable measure and integrals for Archimedean $t-$conorms, J. Math. Anal. Appl. 101 (1984), 114-138.