CONVEXITY OF FUZZY COALITION GAMES*

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Summary

The deterministic coalition game theory is based on the presumption that all expected incomes of coalitions and individual players are exactly known even befor the bargaining process. If we leave this presumption then we extend the original game model into the fuzzy coalition game with vague, i. e., fuzzy, pay-offs. The vagueness of the pay-offs leads to vagueness of some other concepts and properties like the superadditivity and convexity of the game or its core. In this note we formulate the concept of the convexity of fuzzy coalition games and discuss its properties including the convexity of the game and the non-emptiness of its core, known from the deterministic coalition games theory. The type of coalition games dealt in this contribution are the games with side-payments.

Keywords: Game, Coalition game, Core of coalition game, Fuzzy quantity, Fuzzy coalition game, Convex game

1 COALITION GAME WITH SIDE–PAYMENTS

The model of the coalition game with side-payments whose fuzzified modification is investigated here is dealt in many works. Let us remember, as a representation of them, e. g., [8] and [9]. Due to this model, every *coalition game with side-payments* is defined as a pair (I, v), where I is the non-empty and (usually) finite *set of players* and $v : 2^I \to R$, a mapping connecting each set $K \subset I$ with a real number $v(K)$, is called *characteristic function* of the game. Subsets K of I are called *coalitions*. It is natural to put $v(\emptyset) = 0$. Value $v(K)$ represents the expected income of the coalition K.

We say that the game (I, v) is *superadditive* iff for every pair of disjoint coalitions K, $L \subset I$

$$v(K \cup L) \geq v(K) + v(L).$$

Any real-valued vector $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$ is called an *imputation*. The set of imputations C such that

$$C = \Big\{ \boldsymbol{x} \in R^I : \textstyle\sum_I x_i \leq v(I) \text{ and } \sum_K x_i \geq v(K) \text{ for all } K \subset I \Big\} \tag{1}$$

is called the *core of the game* (I, v).

The core, generally, need not be non-empty. A well known sufficient condition for its non-emptiness is the convexity. We say, that (I, v) is *convex* iff for every pair of coalitions K, $L \subset I$

$$v(K \cup L) + v(K \cap L) \geq v(K) + v(L). \tag{2}$$

It is easy to see that the convexity implies the superadditivity of the game.

Due to a well known result, remembered, e. g., in [8, 9], every convex coalition game with side-payments has non-empty core (1).

2 FUZZY QUANTITY

The main fuzzy set theoretical tool used in the following sections for the fuzzification of the coalition game model is the concept of fuzzy quantity. Let us remember, at least briefly, those of its properties used in this paper.

Every fuzzy subset a of the real line R with membership function $\mu_a : R \to [0, 1]$ such that

$$\exists\, x_a \in R : \mu_a(x_a) = 1, \tag{3}$$

$$\exists\, x_1,\, x_2 \in R : x_1 < x_2,\ \mu_a(x) = 0 \text{ for all } x \notin [x_1, x_2], \tag{4}$$

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is called *fuzzy quantity* and the number x_a in (3) is its *modal value*. The fuzzy quantities can be processed by algebraical methods summarized, e. g., in [2] and [4]. In this paper we need to remember the operation of summation and the ordering relation.

Let a, b be fuzzy quantities with membership functions μ_a, μ_b, respectively. Then the sum $a \oplus b$ is also a fuzzy quantity and its membership function $\mu_{a\oplus b}$ is defined by

$$\mu_{a\oplus b}(x) = \sup_{y\in R} \left(\min(\mu_a(y),\, \mu_b(x-y))\right), \quad x \in R. \tag{5}$$

There exist many approaches to the ordering relation over fuzzy quantities which can be found in the accessible literature (e. g., in [3] and [2, 4]). Here, we use the one which is based on the presumption that relation between vague (fuzzy) objects is necessarily also vague being valid with some possibility. If a, b are fuzzy quantities then the possibility that they fulfil weak ordering relation $a \succsim b$ is denoted by $\nu_{\succsim}(a,b) \in [0,1]$ and defined by

$$\nu_{\succsim}(a,b) = \sup_{\substack{x,y\in R\\ x\geq y}} \left(\min(\mu_a(x),\, \mu_b(y))\right). \tag{6}$$

It is evident that $\nu_{\succsim} = 1$ if $x_a \geq x_b$ and that for any real number $r \in R$ the relations $r \succsim a$ and $a \succsim r$ are valid with possibilities
$\nu_{\succsim}(r,a) = \sup(\mu_a(x) : x \in R,\, r \geq x)$,
$\nu_{\succsim}(a,r) = \sup(\mu_a(x) : x \in R,\, x \geq r)$.

3 FUZZY EXTENSION OF COALITION GAME

Let (I,v) be a coalition game and let us suppose that the values $v(K)$, $K \subset I$, are not exactly determined in the time of bargaining and coalition forming. Then it is much more realistic to substitute deterministic values $v(K)$ by some fuzzy quantities reflecting the vagueness of the expectations of the results of cooperation. For every $K \subset I$ we define fuzzy quantity $w(K)$ with membership function $\mu_K : R \to [0,1]$, where for every $K \subset I$, $K \neq \emptyset$,

$$\mu_k(v(K)) = 1, \tag{7}$$

i.e., $v(K)$ is the modal value of $w(K)$,

$$\mu(K) \text{ is increasing for } x < v(K) \tag{8}$$

and decreasing for $x > v(K)$,

and for empty coalition $K = \emptyset$

$$\begin{aligned} \mu_\emptyset(x) &= 1 \quad \text{if} \quad x = v(K), \\ &= 0 \qquad\quad x \neq v(K). \end{aligned} \tag{9}$$

The pair (I,w) is called *fuzzy extension* of the game (I,v) or *fuzzy coalition game*.

Some properties of (I,w) are investigated in [5, 6, 7]. As the expected values of incomes are vague, some other concepts derived from them have to be vague, as well. It regards also the core and the convexity concept.

The core of (I,w) is defined as fuzzy subset C_F of R^I with membership function $\gamma_C : R^I \to [0,1]$, such that for $\boldsymbol{x} = (x_i)_{i\in I} \in R^I$

$$\gamma_C(\boldsymbol{x}) = \min_{K\subset I} \left(\nu_{\succsim}(w(I), \textstyle\sum_I x_i), \nu_{\succsim}(\sum_K x_i,\, w(K)) \right). \tag{10}$$

The possibility that there exists a non-empty core C_F in the fuzzy game (I,w) will be denoted by

$$\gamma_C(I,w) = \sup\left(\gamma_C(\boldsymbol{x}) : \boldsymbol{x} \in R^I\right). \tag{11}$$

4 FUZZY SUPERADDITIVITY AND CONVEXITY

The important properties of superadditivity and convexity can be extended to fuzzy coalition games. We say that (I,w) is *fuzzy superadditive* if for any pair of disjoint coalitions K, $L \subset I$

$$w(K \cup L) \succsim w(K) \oplus w(L). \tag{12}$$

Relation (12) for a given pair K, L is valid with possibility

$$\overline{\nu}(K,L) = \nu_{\succsim}\ (w(K\cup L),\, w(K) \oplus w(L)) \tag{13}$$

due to (6) and, consequently, the possibility that (I,w) is fuzzy superadditive is

$$\sigma(I,w) = \min\left(\overline{\nu}(K,L) : K,\, L \subset I,\, K \cap L = \emptyset\right). \tag{14}$$

Analogously, we say that (I,w) is *fuzzy convex* iff for any pair of coalitions K, $L \subset I$

$$w(K\cup L) \oplus w(K \cap L) \succsim w(K) \oplus w(L) \tag{15}$$

which relation is valid with possibility

$$\nu^*(K,L) = \nu_{\succsim}(w(K\cup L) \oplus w(K\cap L),\, w(K) \oplus w(L)), \tag{16}$$

due to (6), and, consequently, the possibility that (I,w) is fuzzy convex is

$$\delta(I,w) = \min\left(\nu^*(K,L) : K,\, L \subset I\right). \tag{17}$$

Proposition 1. It is evident that for disjoint $K,\ L \subset I$ the equality $\overline{\nu}(K, L) = \nu^*(K, L)$ holds and, consequently, $\delta(I, w) \leq \sigma(I, w)$.

P r o o f. The statement follows from definitions, immediately. □

Proposition 2. If (I, v) is convex then $\gamma_C(I, w) = 1$.

P r o o f. The convexity of (I, v) means that there exists its non-empty core C and this, due to (6), (10), (11) implies $\gamma_C(I, w) = 1$. □

The previous result can be extended on more general cases.

Proposition 3. If (I, w) is fuzzy extension of (I, v) with $\mu_K(x)$ strictly increasing for $x < v(K)$ and strictly decreasing for $x > v(K)$, for any $K \subset I$, and x such that $\mu_K(x) \neq 0$, then

$$\gamma_C(I, w) \geq \delta(I, w).$$

P r o o f. Let (I, w) be fuzzy convex with possibility $\delta(I, w)$. Then there exist real values $v^*(K)$ for each $K \subset I$, such that $\mu_K(v^*(K)) \in [0, 1]$,

$$\min\left(v^*(K) : K \subset I\right) = \delta(I, w)$$

and the deterministic game (I, v^*) is convex. It means that (I, v^*) has a non-empty core C^*. As the fuzzy coalition game (I, w) earns the values of the deterministic game (I, v^*) with the possibility $\delta(I, w)$ then the set C^* is a subset of C_F at least with the same possibility and, consequently, $\gamma_C(I, w)$ is at least equal to $\delta(I, w)$. □

5 AN EXAMPLE

To illustrate the above concepts and propositions we can consider a coalition game with side-payments (I, v) with 3 players such that $I = \{1, 2, 3\}$, and for all $i,\ j = 1, 2, 3,\ i \neq j,\ v(\{i\}) = 1/5,\ v(\{i, j\}) = 2,\ v(I) = 2$. The game (I, v) is neither superadditive (e. g. $v(\{1\}) + v(\{2, 3\}) > v(I)$), nor convex. Its deterministic core $C = \emptyset$. Let us consider fuzzy extension (I, w) of (I, v) such that for $i,\ j \in I,\ i \neq j,\ x \in R$,

$$\begin{aligned} \mu_{\{i\}}(x) &= x + 4/5 \quad \text{for} \quad x \in [-4/5,\, 1/5], \\ &= 6/5 - x \quad \text{for} \quad x \in [1/5,\, 6/5], \\ &= 0 \quad \text{else,} \end{aligned}$$

$$\begin{aligned} \mu_{\{i,j\}}(x) = \mu_I(x) &= x - 1 \quad \text{for} \quad x \in [1, 2], \\ &= 3 - x \quad \text{for} \quad x \in [2, 3], \\ &= 0 \quad \text{else.} \end{aligned}$$

Then it is easy to verify that for $i,\ j,\ k \in I,\ i \neq j \neq k \neq i$

$$\nu_{\succsim}\ (w(\{i, j\}),\ w(\{i\}) \oplus w(\{j\})) = 1$$

$$\nu_{\succsim}\ (w(I),\ w(\{i, j\}) \oplus w(\{k\})) = 14/15$$

and, consequently, (I, w) is superadditive with possibility

$$\sigma(I, w) = 14/15.$$

analogously, for $i,\ j,\ k \in I,\ i \neq j \neq k \neq i$,

$$\nu_{\succsim}\ (w(I) \oplus w(\{j\}),\ w(\{i, j\}) \oplus w(\{j, k\})) = 11/20$$

where, evidently, the membership functions of $w(I) \oplus w(\{j\})$ and $w(\{i, j\}) \oplus w(\{j, k\})$ are $\mu_{I,j},\ \mu_{(i,j)(j,k)}$, respectively, where

$$\begin{aligned} \mu_{I,j}(x) &= (5x - 1)/10 \quad \text{for} \quad x \in [1/5,\, 11/5], \\ &= (21 - 5x)/10 \quad \text{for} \quad x \in [11/5,\, 21/5], \\ &= 0 \quad \text{else,} \end{aligned}$$

$$\begin{aligned} \mu_{(i,j)(j,k)}(x) &= (x - 2)/2 \quad \text{for} \quad x \in [2, 4], \\ &= (6 - x)/2 \quad \text{for} \quad x \in [4, 6], \\ &= 0 \quad \text{else.} \end{aligned}$$

It means that (I, w) is fuzzy convex with possibility

$$\delta(I, w) = 11/20.$$

Finally, the fuzzy core C_F of (I, w) is non-empty with possibility

$$\gamma(I, w) = 3/5.$$

This value is achieved for the imputation $\boldsymbol{x} = (4/5,\, 4/5,\, 4/5)$ which can be realized by the coalition I with the possibility $\mu_I(12/5) = 3/5$ as

$$\nu_{\succsim}\ (w(I),\ 12/5) = \mu_I(12/5),$$

and for any other $\boldsymbol{y} \in R^3$ either $\mu_I(\boldsymbol{y}) < 3/5$ if $\boldsymbol{y} > \boldsymbol{x}$, or for some 2-players coalition $\{i, j\}$

$$\begin{aligned} &\nu_{\succsim}\ (y_i + y_i,\ w(\{i, j\})) \\ < \ &\nu_{\succsim}\ (3/5 + 3/5,\ w(\{i, j\})) = \mu_{\{i,j\}}(8/5) = 3/5. \end{aligned}$$

Note, that

$$\gamma(I, w) \geq \delta(I, w).$$

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