BALANCING PROPERTY OF FUZZY MEASURES

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Summary

In this work, we first introduce a balancing property on a fuzzy measure. After that, we give the conditions of when an additive fuzzy measure satisfies the balancing property and similar results are presented for 0-1 and S–decomposable fuzzy measures where S is a t–conorm. Moreover, we introduce the concept of distance between two additive fuzzy measures and develop some results related to the distance and the balancing property.

Keywords: fuzzy measures, balancing property, discrete Choquet and Sugeno integrals, aggregation operator, OWA operator, weighted mean.

1 INTRODUCTION

Fuzzy measures and integrals have been introduced by Sugeno in 1974 [17] in order to generalize probability measures. Since that time, they have been studied from different points of view. Particularly, in recent years, there has been an increasing interest in discrete fuzzy integrals, and specially in discrete Choquet and Sugeno integrals, both from a theoretical and an applicational point of view (see e.g. [4, 5, 6, 12, 14, 15, 16]).

On the other hand, another field of increasing interest is related to the aggregation problem. In particular, a lot of authors are interested in the so called aggregation operators. Both problems are closely related because discrete fuzzy integrals can be used as new aggregation tool since they define particular classes of continuous idempotent aggregation operators. It is from the aggregation point of view that we are interested in fuzzy measures and so we use only basic definitions restricted to the discrete case. For more details, we refer to [4, 5, 15, 18].

The purpose of this work is to present a survey of recent results on a subclass of discrete fuzzy measures, i. e., the class of discrete fuzzy measures that satisfy the property called: *balancing property*. The idea of this property is to assign a larger measure to a set of criteria than to any other set with less criteria. As we see along the paper, these results show clearly the situation of fuzzy integrals among aggregation operators, from a theoretical point of view.

2 PRELIMINARIES

In this section, we recall some basic definitions that we shall use throughout this paper. From now on, let us consider a finite universe $X = \{1, \ldots, n\}$.

Definition 1 [17] A fuzzy measure μ on X is a $\mathcal{P}(X) \longrightarrow [0, 1]$ mapping that satisfies:

(i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$,

(ii) monotonicity: $A \subset B \Rightarrow \mu(A) \leq \mu(B)$, for all $A, B \subseteq X$.

Definitions on more general spaces usually require algebras and σ–algebras, but this is not necessary in the discrete case. A fuzzy measure on X needs $2^n - 2$ coefficients to be defined, which are the values of μ for all different subsets of X different from X and $\emptyset$.

It is said that a fuzzy measure is

1. a 0–1 *fuzzy measure* if it takes only 0 and 1 as value,
2. *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$,
3. *symmetric* if the following implication holds, for any subsets A and B of X:

$$|A| = |B| \Rightarrow \mu(A) = \mu(B),$$

4. *maxitive* if $\mu(A \cup B) = \mu(A) \vee \mu(B)$,
5. *S–decomposable* measure if $\mu(A \cup B) = S(\mu(A), \mu(B))$ whenever $A \cap B = \emptyset$, with S a continuous t–conorm.

For any fuzzy measure μ, the *dual measure* μ^* is defined by $\mu^*(A) = 1 - \mu(A^c)$ for any $A \subseteq X$. We can quote that if μ is additive then $\mu = \mu^*$, but the converse is not true.

Let us also recall the definitions of discrete Choquet and Sugeno integrals directly as operators from $[0,1]^n$ to $[0,1]$. From now on, given n numbers $x_1, \dots, x_n$, we will denote by $x_{(1)}, \dots, x_{(n)}$ the permutation arranged in increasing order: $x_{(1)} \leq \cdots \leq x_{(n)}$.

Definition 2 Given a fuzzy measure μ, the n-dimensional discrete Sugeno integral with respect to μ is the function $S_\mu : [0,1]^n \to [0,1]$ defined by

$$S_\mu(x_1, \dots, x_n) = \bigvee_{i=1}^{n} (x_{(i)} \wedge \mu(A_{(i)}))$$

where $A_{(i)} = \{(i), (i+1), \dots, (n)\}$.

Definition 3 Given a fuzzy measure μ, the n-dimensional discrete Choquet integral with respect to μ is the function $C_\mu : [0,1]^n \to [0,1]$ defined by

$$C_\mu(x_1, \dots, x_n) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)})\, \mu(A_{(i)})$$

with the same notations as above, and $x_{(0)} = 0$.

Remark 1 Observe that $A_{(n)} \subset A_{(n-1)} \subset \cdots \subset A_{(2)} \subset A_{(1)} = X$, and $|A_{(i)}| = n - i + 1$, for all $i = 1, \dots, n$.

The $[0,1]^n \to [0,1]$ mappings C_μ and S_μ that are obtained in this way are always continuous and idempotent aggregation operators. The main results on the relation between fuzzy integrals and aggregation operators are summarized below (see [5, 6, 15]):

- A discrete Choquet integral with respect to an additive fuzzy measure coincides with a weighted arithmetic mean.
- The discrete Choquet integral reduces to the OWA operator for every symmetric fuzzy measure.
- The discrete Sugeno integral corresponds to a weighted median (Kandel and Byatt (1978))

$$S_\mu(x_1, \dots, x_n) =$$
$$= med(x_1, \dots, x_n, \mu(A_{(2)}), \dots, \mu(A_{(n)})).$$

- If μ is a $0-1$ fuzzy measure, then the Choquet and Sugeno integrals are the same aggregation function, which takes the following form:

$$\begin{aligned} C_\mu(x_1, \dots, x_n) &= S_\mu(x_1, \dots, x_n) = \\ &= \bigvee_{A:\mu(A)=1} \left(\wedge_{i \in A} x_{(i)}\right). \end{aligned} \quad (1)$$

These are the only continuous aggregation functions verifying $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$ [11].

3 BALANCING PROPERTY OF A FUZZY MEASURE

Definition 4 A set function $\mu : \mathcal{P}(X) \to [0,1]$ satisfies the *balancing property* if

$$|A| < |B| \Rightarrow \mu(A) \leq \mu(B) \quad (2)$$

Proposition 1 *Let $\mu : \mathcal{P}(X) \to [0,1]$ be a set function verifying the balancing property.*

(i) If $\mu(\emptyset) = 0$ and $\mu(X) = 1$ then μ is a fuzzy measure and it will be called a BP–fuzzy measure.

(ii) Any symmetric fuzzy measure is a BP–fuzzy measure.

Remark 2 Obviously, there are fuzzy measures such that they are not the BP–fuzzy measures. Note also that a BP–fuzzy measures need not be symmetric.

Proposition 2 *μ is a BP–fuzzy measure if and only if its dual μ^* is also a BP–fuzzy measure.*

3.1 BALANCING PROPERTY OF A 0-1 FUZZY MEASURE

It is possible to determine 0-1 fuzzy measures that satisfy the balancing property and also the aggregation functions defined by this class of measures.

Example 1 For $n = 3$, there are eighteen 0-1 fuzzy measures:

1. Three of them correspond to projections, which are also called Dirac measures. They are the only 0-1 fuzzy measures such that are not BP–fuzzy measures.
2. Three of them correspond to the order statistics, $S_3^1 = \min$, $S_3^2 = med$ and $S_3^3 = \max$. They are the only symmetric 0-1 fuzzy measures.
3. The other 12 measures are not symmetric measures but they are BP–fuzzy measures.

This clasification does not hold for $n > 4$: there are measures that are not BP–fuzzy measures apart from projections.

However, the following result characterizes them, and gives its number:

Proposition 3 *Let μ be a 0-1 BP–fuzzy measure defined on $X = \{1, \ldots, n\}$.*

(i) *There exists s, $0 < s < n$, such that for any $A \subset X$, if $|A| < s$ then $\mu(A) = 0$; if $|A| > s$ then $\mu(A) = 1$ and if $|A| = s$ then $\mu(A) = 0$ or 1.*

(ii) *There are $\sum_{i=1}^{n} 2^{\binom{n}{i}} - n$ of these measures.*

The corresponding discrete integrals can be obtained via (1).

3.2 BALANCING PROPERTY OF AN ADDITIVE FUZZY MEASURE

Here we study the additive measures that verify the balancing property and their corresponding Choquet integrals, the weighted arithmetic means. First, we observe that if μ is additive then it is a discrete probability measure and it is uniquelly determined by the probability vector $(p_1, \ldots, p_n)$, $p_i = \mu(\{(i)\})$. The corresponding Choquet integral coincides with the standard Lebesgue integral and it results to the weighted mean, $C_\mu = \sum_{i=1}^{n} x_i p_i$.

Given a probability vector $p = (p_1, \ldots, p_n)$, let $p^\star = (\alpha_1, \ldots, \alpha_n)$ be a non–decreasing permutation of p.

Proposition 4 *Let μ be an additive fuzzy measure defined on $X = \{x_1, \ldots, x_n\}$.*

(i) *If $n = 2$, then μ is a BP–fuzzy measure.*

(ii) *For $n \geq 3$, μ is a BP–fuzzy measure if and only if*

$$\sum_{k=1}^{r-1} \alpha_{n-k+1} \leq \sum_{k=1}^{r} \alpha_k, \quad r = [\frac{n+1}{2}]. \tag{3}$$

The above inequality (3) means that the smallest measure of a subset with cardinality r is greater or equal to the largest measure of a subset whit cardinality $r - 1$.

For $n = 3$, the above property is $\alpha_3 \leq \alpha_1 + \alpha_2$; for $n = 4$, $\alpha_4 \leq \alpha_1 + \alpha_2$; for $n = 5$, $\alpha_5 + \alpha_4 \leq \alpha_1 + \alpha_2 + \alpha_3$, and so on.

Example 2 We consider a weighted mean with weights $(0.250, 0.335, 0.165, 0.250)$ to give a global evaluation of four different proves. Here, $n = 4$, $r = 2$ and $\alpha_4 = 0.335 \leq \alpha_1 + \alpha_2 = 0.415$, hence condition (2) is satisfied. We can say that they provide a *balancing* between the marks of all proves since the mark of one proof has less significance than the sum of two other marks.

Now, we define a distance between additive measures.

Definition 5 Let μ_1 and μ_2 be two additive fuzzy measures determined by probability vectors $p_1 = (p_1^{(1)}, \ldots, p_n^{(1)})$ and $p_2 = (p_1^{(2)}, \ldots, p_n^{(2)})$. Then the distance between μ_1 and μ_2 is defined by:

$$d(\mu_1, \mu_2) = \sum_{i=1}^{n} |p_i^{(1)} - p_i^{(2)}|. \tag{4}$$

Note that d is the restriction of the standard L_1-distance in R^n.

Let μ_a be the measure corresponding to the arithmetic mean M_a, i. e., μ_a is an uniformly distributed probability measure defined by the probability vector $p = (\frac{1}{n}, \ldots, \frac{1}{n})$. Then $d(\mu_1, \mu_a) = \sum_{i=1}^{n} |p_i^{(1)} - \frac{1}{n}| = \sum_{i=1}^{n} |\alpha_i - \frac{1}{n}|$, where $(\alpha_1, \ldots, \alpha_n) = p_1^\star$.

Lemma 1 *The distance between any additive measure μ_1 and μ_a is given by*

$$d(\mu_1, \mu_a) = \frac{2q}{n} - 2 \cdot \sum_{i=1}^{q} \alpha_i, \tag{5}$$

where q is the last index such that $\alpha_q \leq \frac{1}{n}$.

Let us see now how additive BP–fuzzy measures must be.

Proposition 5 *If an additive measure μ is a BP–fuzzy measure then it musts satisfy that:*

$$d(\mu, \mu_a) \leq \frac{2}{n}. \tag{6}$$

The converse does not hold. Not any measure verifying (6) is a BP–fuzzy measure as we show in the next result:

Lemma 2 *An additive BP–fuzzy measure μ with probability vector p satisfies $d(\mu, \mu_a) = \frac{2}{n}$ if and only if $p^\star = (0, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$.*

There are also additive measures in the interior of $B(\mu_a, \frac{2}{n}) = \{\mu \mid d(\mu_a, \mu) \leq \frac{2}{n}\}$, which are not BP–fuzzy measures. However we get the following result:

Proposition 6 *Let $\mu \in B(\mu_a, \frac{1}{n})$ be an additive measure. Then μ is a BP–fuzzy measure. Moreover, for any $\epsilon > 0$ there exist an additive measure $\mu \in B(\mu_a, \frac{1}{n} + \epsilon)$ which is not a BP–fuzzy measure.*

3.3 BALANCING PROPERTY OF S–DECOMPOSABLE FUZZY MEASURES

In this section, we consider a continuous t–conorm S and analogously to the Section 3.2 it can be shown

Proposition 7 *An S–decomposable fuzzy measure μ is a BP–fuzzy measure if and only if*

1. $\alpha_n = 1 = S(\alpha_1, \alpha_2)$.
2. $\alpha_n < 1$ *and for* $p = \min(r, q-1)$ *it holds*

$$S(\alpha_1, \ldots, \alpha_p) \geq S(\alpha_{n-p+1}, \ldots, \alpha_n), \qquad (7)$$

with $q = \min\{i \in \{2, \ldots, n\} | S(\alpha_1, \ldots, \alpha_i) = 1\}$, $r = [\frac{n+1}{2}]$ *and* $n \geq 3$. *We here consider that* $S(x) = x$ *for all* x *in* $[0,1]$.

Note that $\alpha_n < 1$ immediatly means that S cannot be a t–conorm such that $S(x,y) = 1$ only if $\max(x,y) = 1$ (i.e., S is either a nilpotent t–conorm, or an ordinal sum with nilpotent summand on the interval $[a,1], a$ in $[0,1)$).

Proposition 8 *If* $S(x,y) = 1$ *only when* $\max(x,y) = 1$, *then an S–decomposable fuzzy measure μ is a BP–fuzzy measure if and only if* $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 1$ *and* $\alpha_1 \in [0,1]$. *Furthermore,*

$$S_\mu(x_1, \ldots, x_n) = \max(x_1, \ldots, x_i \wedge \alpha_1, \ldots, x_n),$$

where $\mu(\{i\}) = \alpha_1$.

In this case, the corresponding Sugeno integral defined by an S–decomposable fuzzy measure is close to the maximum operator.

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