EMBEDDING SOME FAMILIES OF FUZZY RELATIONS IN THE CLASS OF FUZZY CONSEQUENCE OPERATORS.

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Abstract

The aim of this paper is to show several isomorphic structures between fuzzy relations and operators from the family of fuzzy sets into itself. In particular, we prove that the $*$-preorders family is able to embed itself in the class of fuzzy consequence operators by way of a lattice monomorphism.

Keywords : Fuzzy Consequence Operator, Closure Operator, Fuzzy Preorder, Fuzzy Logic, Lattice.

1 INTRODUCTION

Since L. Zadeh introduced Fuzzy Set Theory ([13], 1965), the use of the $*$-preorder concept is well-known.

Given X a nonempty universal set and given a t-norm $*$, a fuzzy relation on X (fuzzy subset of $X \times X$) is called $*$-fuzzy preorder if it verifies :
(R1) $R(x,x) = 1 \ \forall\ x \in X$ (reflexivity)
(R2) $R(x,z) \geq R(x,y) * R(y,z) \ \forall\ x,y,z \in X$ ($*$-transitivity).

In this paper Γ', Γ_r and Γ^* will represent the class of fuzzy relations on X, the family of fuzzy reflexive relations and the subfamily of fuzzy $*$-preorders, respectively.

In fuzzy logic, the fuzzy consequence operator concept is relevant, FCO abbreviately. Given L a closed lattice, a function $C : L^X \longrightarrow L^X$ is called FCO (J. Pavelka, [10]) if it verifies :
(C1) $\mu \subseteq C(\mu) \ \forall\ \mu \in L^X$ (inclusion)
(C2) $\mu_1 \subseteq \mu_2 \Longrightarrow C(\mu_1) \subseteq C(\mu_2) \ \forall\ \mu_1,\ \mu_2 \in L^X$ (monotony)
(C3) $C(C(\mu)) = C(\mu) \ \forall\ \mu \in L^X$ (idempotence).

These operators, sometimes called closure operators, are studied in a general context ([12]). During the last decade, these operators have been also studied in the context of fuzzy logic, taking the chain $L = [0,1]$ as a special case ([2],[3],[4],[6],[7],[11]). In this paper we will consider $L = [0,1]$.

We will denote by Ω' the class of operators from $[0,1]^X$ into $[0,1]^X$, Ω will represent the family of fuzzy consequence operators and for every t-norm $*$, Ω_p^* will represent the subfamily of fuzzy consequence operators which are induced by a $*$-preorder through Zadeh's max-$*$ compositional rule, this is :
$\Omega_p^* = \{C \in \Omega \mid \exists R \in \Gamma^*, C = C_R^*\}$, where $C_R^*(\mu) = \mu \circ^* R : C_R^*(\mu)(x) = \sup_{w \in X} \{\mu(w) * R(w,x)\}$.

Finally, we recall some notions and results on general lattice theory ([1],[9]) :

Given $(L, \leq, \vee, \wedge)$ a lattice and given S a subset of L, we will say that S is a sublattice of L if $(S, \leq, \vee, \wedge)$ is a lattice, this is, if it verifies :
$a \vee b \in S \ \forall\ a,b \in S$ (1) and $a \wedge b \in S \ \forall\ a,b \in S$ (2).
In this case we will write $S \leq L$. If only (1) is verified we will say that S is a $\vee$-sublattice (join-sublattice) of L and we will write $S \leq_\vee L$. If only (2) is verified we will say that S is a $\wedge$-sublattice (meet-sublattice) of L and we will write $S \leq_\wedge L$.

Let $\alpha : L \longrightarrow M$ be a function from a lattice L into a lattice M. We will say that α is isotone when $a \leq b$ implies $\alpha(a) \leq \alpha(b)$. We will say that α is a $\vee$-morphism (join-morphism) if $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ for all $a, b \in L$; a $\wedge$-morphism (meet-morphism) if $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$ for all $a, b \in L$; and a morphism when both conditions hold. As always, a morphism is called a monomorphism if it is one-one; an epimorphism if onto and an isomorphism if a bijection. If there exists an isomorphism between L and M, we will write $L \cong M$; if there exists a join-isomorphism, $L \cong_\vee M$ and if a meet-isomorphism, $L \cong_\wedge M$.

The following result holds : Any join-morphism between lattices is isotone; also any meet-morphism is isotone; and finally, any isotone bijection with isotone inverse is a lattice isomorphism.

2 LATTICE SUBSTRUCTURES

It is well-known (J.A. Goguen, [8]) that if L is a distributive complete lattice then the functional space L^X has the same structure.

As $([0,1], \leq, \vee, \wedge)$ is a distributive complete lattice (in fact, it is a chain), so is the family $[0,1]^X$ of fuzzy subsets of X with the following natural operations :

$$\mu_1 \subseteq \mu_2 \iff \mu_1(x) \leq \mu_2(x) \ \forall\ x \in X$$
$$(\mu_1 \vee \mu_2)(x) = \max\left(\mu_1(x), \mu_2(x)\right)$$
$$(\mu_1 \wedge \mu_2)(x) = \min\left(\mu_1(x), \mu_2(x)\right).$$

The family $\Gamma' = [0,1]^{X \times X}$ as family of fuzzy subsets of $X \times X$ is also a distributive complete lattice with the previous operations between fuzzy relations.

Analogously the family $\Omega' = \left([0,1]^X\right)^{[0,1]^X}$ is another distributive complete lattice with the operations :

$$C_1 \subseteq C_2 \iff C_1(\mu) \leq C_2(\mu) \ \forall\ \mu \in [0,1]^X$$
$$(C_1 \vee C_2)(\mu)(x) = \max\left(C_1(\mu)(x), C_2(\mu)(x)\right)$$
$$(C_1 \wedge C_2)(\mu)(x) = \min\left(C_1(\mu)(x), C_2(\mu)(x)\right).$$

On the other hand, it is clear that $\Gamma_r \leq \Gamma'$ and we proved outside this work that $\Gamma^* \leq_\wedge \Gamma'$ and $\Omega_p^* \leq_\wedge \Omega'$ but $\Gamma^* \nleq_\vee \Gamma'$ and $\Omega_p^* \nleq_\vee \Omega'$, for every t-norm $*$. Thus:

$$\Gamma^* \leq_\wedge \Gamma_r \leq \Gamma' \quad \text{and} \quad \Omega_p^* \leq_\wedge \Omega \leq \Omega'.$$

3 ISOMORPHIC STRUCTURES

For every t-norm $*$, we consider the function $\theta'^* : \Gamma' \longrightarrow \Omega'$ given by $\theta'^*(R) = C_R^*$ for any fuzzy relation R, where $C_R^*(\mu)(x) = \sup_{w \in X} \left\{\mu(w) * R(w,x)\right\}$. θ^* will represent the restriction of θ'^* to the family Γ^*. We also consider the function $\tilde{\theta}' : \Omega' \longrightarrow \Gamma'$ given by $\tilde{\theta}'(C) = R_C$ for any operator C, where $R_C(x,y) = C(\varphi_x)(y)$ and φ_x represents the crisp membership of the singleton $\{x\}$. We can note that the function $\tilde{\theta}'$ does not depend on the t-norm $*$.

The following result holds for any fuzzy relation R. This result is proved by J.L. Castro and E. Trillas ([4]) if we suppose that R is a fuzzy $*$-preorder.

Theorem 1.- *For any fuzzy relation R on X and for any t-norm $*$, $R_{C_R^*}$ is exactly the relation R.*

Proof.- $R_{C_R^*}(x,y) = C_R^*(\varphi_x)(y)$. But

$$C_R^*(\varphi_x)(y) = \sup_{w \in X} \left\{\varphi_x(w) * R(w,y)\right\}$$

If $w \neq x$, $\varphi_x(w) = 0$. Hence this supremum is taked in $w = x$ and it is equal to

$$\varphi_x(x) * R(x,y) = 1 * R(x,y) = R(x,y) \qquad \square$$

As a consequence :

Corollary 1.- *$\tilde{\theta}' \circ \theta'^*$ results the identity mapping on Γ'.*

Corollary 2.- *The function $\theta'^* : \Gamma' \longrightarrow \Omega'$ is one to one.*

Proof.- If $\theta'^*(R_1) = \theta'^*(R_2)$, $\tilde{\theta}' \circ \theta'^*(R_1) = \tilde{\theta}' \circ \theta'^*(R_2)$ and from corollary 1, $R_1 = R_2$. $\square$

We will denote by $\Omega_p'^*$ the set $\Im m(\theta'^*)$. It is clear that $\Omega_p'^* \subsetneq \Omega'$ and thus θ'^* is not onto. In fact :

Theorem 2.- $\Omega_p'^* = \left\{C \in \Omega' \mid \theta'^* \circ \tilde{\theta}'(C) = C\right\}$.

Proof.- If $C \in \Im m(\theta'^*)$, there exists a unique $R \in \Gamma'$ such that $\theta'^*(R) = C$. From corollary 1, $\tilde{\theta}' \circ \theta'^*(R) = R$ and $\theta'^* \circ \tilde{\theta}'(C) = (\theta'^* \circ \tilde{\theta}') \circ \theta'^*(R) = \theta'^* \circ (\tilde{\theta}' \circ \theta'^*)(R) = \theta'^*(R) = C$. Conversely, if $\theta'^* \circ \tilde{\theta}'(C) = C$, C is the image of the fuzzy relation $\tilde{\theta}'(C)$ by the mapping θ'^* and thus $C \in \Im m(\theta'^*)$. $\square$

Theorem 3.- *The function $\theta'^* : \Gamma' \longrightarrow \Omega_p'^*$ is a lattice isomorphism. Thus : $\Gamma' \cong \Omega_p'^*$ and $\Omega_p'^* \leq \Omega'$.*

Proof.- θ'^* is obviously an isotone bijection. But ${\theta'^*}^{-1} : \Omega_p'^* \longrightarrow \Gamma'$ is exactly the restriction of $\tilde{\theta}'$ to $\Omega_p'^*$ which is also isotone. Therefore θ'^* is a lattice isomorphism. $\square$

Accordingly, we recall that the function θ'^* is a lattice morphism : $\theta'^*(R_1 \vee R_2) = \theta'^*(R_1) \vee \theta'^*(R_2)$ and $\theta'^*(R_1 \wedge R_2) = \theta'^*(R_1) \wedge \theta'^*(R_2)$ for all fuzzy relations R_1, R_2 .

In particular, we have for all $*$-preorders R_1, R_2 : $\theta'^*(R_1 \vee R_2) = \theta^*(R_1) \vee \theta^*(R_2)$ and $\theta^*(R_1 \wedge R_2) = \theta^*(R_1) \wedge \theta^*(R_2)$, where θ^* is a bijection between Γ^* and Ω_p^* with inverse isotone.

However, since Γ^* is not a sublattice of Γ' we cannot say that there exists an isomorphism between Γ^* and Ω_p^* with the usual lattice operations. But it is possible to modify the previous operations to obtain an isomorphism between both families. We will analyze this question at the end of this section.

At all events, we have the following representation theorems :

Theorem 4.- *The function* $\theta^* : \Gamma^* \longrightarrow \Omega_p^*$ *is a meet-isomorphism. Thus :* $\Gamma^* \cong_\wedge \Omega_p^* \leq_\wedge \Omega \leq_\wedge \Omega'$.

Theorem 5.- *The restricted function* $\theta'^* \mid \Gamma_r : \Gamma_r \longrightarrow \Omega'$ *is a lattice monomorphism. Thus :*
$\Gamma_r \cong \Omega'^*_{pi}$, *where* $\Omega'^*_{pi} = \Im m_{\theta'^*}(\Gamma_r)$, *this is :*
$\Omega'^*_{pi} = \{C \in \Omega_p'^* \mid C \text{ verifies the inclusion axiom}\}$.

We will show a last isomorphism with the usual operations, extending minimally the family Γ^* for this purpose :

Let $\overline{\Gamma}^*$ be the sublattice generated by Γ^*, this is, the smallest sublattice of Γ' containing Γ^*. It is clear that $\overline{\Gamma}^* = \Big\{R_1 \vee R_2 \vee \cdots \vee R_n \mid n \in \mathbb{N}, R_i \in \Gamma^*\Big\}$.

Now, we consider $\overline{\theta}^*$ as the restricted function θ'^* to $\overline{\Gamma}^*$. Analogously, $\overline{\theta}^*$ is a lattice monomorphism. Moreover, if we denote by $\overline{\Omega}_p^*$ the set $\Im m(\overline{\theta}^*)$ then $\overline{\Omega}_p^* = \Big\{C_1 \vee C_2 \vee \cdots \vee C_n \mid n \in \mathbb{N}, C_i \in \Omega_p^*\Big\}$ and $\overline{\Omega}_p^*$ result the sublattice of Ω' generated by Ω_p^*. Hence :

Theorem 6.- *The function* $\overline{\theta}^* : \overline{\Gamma}^* \longrightarrow \overline{\Omega}_p^*$ *is a lattice isomorphism. Thus :* $\overline{\Gamma}^* \cong \overline{\Omega}_p^* \leq \Omega'$.

The previous result is a representation theorem of all fuzzy relations which are finite union of fuzzy $*$-preorders by way of the family of operators which are finite union of fuzzy consequence operators.

From now, we will consider $*$ as a **continuous** t-norm with the aim of obtaining an isomorphism between Γ^* and Ω_p^*.

Since Γ^* is not a join-sublattice of Γ' we are obliged to vary the $\vee$ operation for this purpose to be possible.

Given a continuous t-norm $*$, it is proved ([3]) that the family Γ^* of all fuzzy $*$-preorders is a complete lattice with the usual $\subseteq$, $\wedge$ operations and the following $\sqcup^*$:

$$R_1 \sqcup^* R_2 = \bigcap \big\{R \in \Gamma^* \mid R \supseteq R_1 \vee R_2\big\}.$$

Notice that the following properties hold, hence $\sqcup^*$ can work as a union operation :
($\sqcup^*$ 1) $R_1 \sqcup^* R_2 \supseteq R_1 \vee R_2$.
($\sqcup^*$ 2) $R_1 \sqcup^* R_2 = R_1 \vee R_2 \iff R_1 \vee R_2 \in \Gamma^*$.
($\sqcup^*$ 3) $R_1 \sqcup^* R_2$ is the smallest $*$-preorder containing R_1 and R_2.
($\sqcup^*$ 4) $R_1 \sqcup^* R_2 = R_1 \iff R_1 \supseteq R_2$.

The next example proves that $\Big(\Gamma^*, \sqcup^*, \wedge, \subseteq\Big)$ is a **non**-distributive lattice.

Let X be an universe with card$X \geq 3$, we consider the following crisp relations on X :

$$R_1 = \Big\{(x,y),(y,x)\Big\} \cup \Big\{(t,t)\Big\}_{t\in X}$$
$$R_2 = \Big\{(y,z),(z,y)\Big\} \cup \Big\{(t,t)\Big\}_{t\in X}$$
$$R_3 = \Big\{(x,z),(z,x)\Big\} \cup \Big\{(t,t)\Big\}_{t\in X}$$

We have : R_1, R_2, $R_3 \in \Gamma^*$, $R_1 \vee R_3$, $R_2 \vee R_3 \notin \Gamma^*$. So $R_1 \wedge R_2 = \Big\{(t,t)\Big\}_{t\in X} \in \Gamma^*$. On the other hand $R_1 \sqcup^* R_3 = R_2 \sqcup^* R_3 = R_1 \vee R_2 \vee R_3$. Thus :

$$\big(R_1 \sqcup^* R_3\big) \wedge \big(R_2 \sqcup^* R_3\big) = R_1 \vee R_2 \vee R_3 \neq$$

$$\neq R_3 = \big(R_1 \sqcup^* R_2\big) \wedge R_3$$

Now, we define analogously the operation $\sqcup^*$ in Ω_p^*. Given C_1, $C_2 \in \Omega_p^*$:

$$C_1 \sqcup^* C_2 = \bigcap \big\{C \in \Omega_p^* \mid C \supseteq C_1 \vee C_2\big\}$$

This operation is well defined : let R be the trivial relation $R(x,x) = 1$ for all $x \in X$ then for any t-norm $*$, $\theta^*(R)$ is the operator C_{sup} given by

$$C_{sup}(\mu)(t) = \sup_{w\in X} \big\{\mu(w)\big\}.$$

Clearly, C_{sup} is the greatest element of the set Ω_p^*. Hence, given C_1, $C_2 \in \Omega_p^*$, $C_{sup} \supseteq C_1 \vee C_2$ and thus $\big\{C \in \Omega_p^* \mid C \supseteq C_1 \vee C_2\big\}$ is a nonempty set. As Ω_p^* is a complete $\wedge$-sublattice, the operation $\sqcup^*$ is well defined.

The analogous properties ($\sqcup^*$ 1),($\sqcup^*$ 2),($\sqcup^*$ 3),($\sqcup^*$ 4) for $*$-preorders also hold for operators. Hence $\sqcup^*$ also can work as a union operation in Ω_p^*.

It is easy to prove that $\Big(\Omega_p^*, \sqcup^*, \wedge, \subseteq\Big)$ is also a complete **non**-distributive lattice.

Finally, we can show the representation theorem which identifies both structures.

Theorem 7.- *The function* θ^* *is a lattice isomorphism between* $\left(\Gamma^*, \sqcup^*, \wedge, \subseteq\right)$ *and* $\left(\Omega_p^*, \sqcup^*, \wedge, \subseteq\right)$.

Proof.- The bijection θ^* from Γ^* into Ω_p^* is isotone and so its inverse ${\theta^*}^{-1} = \tilde{\theta} \mid \Omega_p^*$. □

4 CONCLUSIONS

It is clear that the functions θ'^* change essentially on t-norm $*$. Consequently, the set $\Omega_p^{'*}$ of operators $*$-induced by relations is different from $\Omega_p^{'\perp}$ if $* \neq \perp$.

Nevertheless, by theorem 3, $\Omega_p^{'*} \cong \Omega_p^{'\perp} \cong \Gamma'$ for all t-norms $*$, $\perp$. Analogously, by theorem 5, $\Omega_{pi}'^{*} \cong \Omega_{pi}'^{\perp} \cong \Gamma_r$.

It is not true for two sets Ω^* and $\Omega^\perp$ of preorders induced fuzzy consequence operators, this is, Ω^* and $\Omega^\perp$ are not isomorphic if $* \neq \perp$.

Moreover, we have shown the minimal isomorphism $\overline{\Gamma}^* \cong \overline{\Omega}_p^*$ with the usual $\vee$, $\wedge$ operations, where $\overline{\Gamma}^*$ is the sublattice generated by the family of $*$-fuzzy preorders.

In the context of fuzzy preorders and fuzzy consequence operators we have only a meet-isomorphism : $\Gamma^* \cong_\wedge \Omega_p^*$.

We had to change the usual union to obtain an isomorphism between Γ^* and Ω_p^*. In fact, we show that θ^* is a lattice isomorphism between the non-distributive lattices $\left(\Gamma^*, \sqcup^*, \wedge, \subseteq\right)$ and $\left(\Omega_p^*, \sqcup^*, \wedge, \subseteq\right)$.

We recall that the class Ω of fuzzy consequence operators is not a sublattice in Ω'. However, Ω form a lattice within Ω' with the usual intersection and the following union ([2],[12]) :

$$C_1 \sqcup' C_2 = \bigcap \left\{ C \in \Omega \mid C \supseteq C_1 \vee C_2 \right\} \ \forall\, C_1, C_2 \in \Omega.$$

We proved recently ([5]) that the restriction of $\sqcup'$ to the family Ω_p^* is precisely $\sqcup^*$. Therefore we have embedded the family of all $*$-preorders Γ^* in the class of fuzzy consequence operators Ω by a lattice monomorphism : $\Gamma^* \cong \Omega_p^* \leq \Omega$.

Finally, since the lattice Ω and the lattice of all closure systems on X (a family of fuzzy subsets of X is called closure system when it is closed under arbitrary intersections and it contains X) are dually isomorphic ([2]), the families Γ^* and Ω_p^* are called to embedding in the structure of closure systems under the dual isomorphism concept.

The purpose of this paper is to complete the relation between fuzzy preorders and FCOs in order to use the FCO concept in approximate reasoning. This question will require further research.

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