FUZZY ARITHMETIC AND FUZZY RANKING: A NONSTANDARD APPROACH

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Summary

In this paper, we analyze relations between nonstandard analysis and fuzzy sets theory. We propose an interpretation of fuzzy sets using a nonstandard approach and we propose a fuzzy ranking method based on this interpretation.

Palabras Clave: Nonstandard Analysis, Ranking of fuzzy numbers..

1 Introduction

Although it seems that Cantorian set theory has prevailed in the whole field of mathematics, there are some alternative theories where the invention of new set-like concepts is inevitable. Drossos summarizes a partial list of such instances. Nonstandard Analysis (Robinson 1961) and Fuzzy Sets (Zadeh 1965) are included in this list.

We may divide these theories into two categories: The extensional and the intentional one. Let A be a set, then any function

$$p : I \to A \tag{1}$$

represents an extensional aspects of A (e.g. a list of elements, a path, etc.). If the index set is $I = \{1, 2, \ldots, n\}$, every application p can be associated with an element of A^n. Nonstandard analysis is another example of extension.

Coming now to intentional aspects of a set A, we consider instead of incoming functions, outgoing ones. That is functions of the type

$$p : A \to I. \tag{2}$$

If the index set is $I = \{1, 2, \ldots, n\}$, we divide the elements of A according to a given property. The definition of a fuzzy subset of A is another example of intention.

2 Nonstandar Analysis Development

The classic way of constructing hyper-real numbers is to consider the natural numbers set N, an ultrafilter U in N (usually an ultrafilter containing Frechets filter) and the real numbers set R. Considering the set of applications from N to R and the equivalence classes given by the equivalence relation

$$f \sim g \quad \text{iff} \quad \{n \in N : f(n) = g(n)\} \in U, \tag{3}$$

the set of the hyper-real numbers is obtained. This set constitutes an ordered field, and it is non-archimedian since it contains numbers which are greater than all numbers of R , which constitutes a subfield of it.

In this paper we consider the construction of the hyper-reals as follows: we use the set $I = [0, 1]$ instead of N and an ultrafilter containing the filter formed by all sets which the Lebesgue's measure is equal to 1. Considering an equivalence relation as in (3)

$$f \sim g \quad \text{iff} \quad \{x \in [0, 1] : f(x) = g(x)\} \in U, \tag{4}$$

we have an ordered field which we call HR.

The addition and multiplication of hyper-real numbers are defined in a natural way as follows:

$$(f + g)(x) = f(x) + g(x) \tag{5}$$

$$(fg)(x) = f(x)g(x) \tag{6}$$

Where the functions defined as

$$f(x) = 0 \quad \forall x \in I \qquad\qquad f(x) = 1 \quad \forall x \in I \tag{7}$$

are, respectively, the neutral elements for addition and multiplication. The associativity and commutativity can be trivially probed for both operators. The inverse element of f under addition is $-f$ and, if f is a non-zero element, the function $1/f$ defined as

$$(1/f)(x) = \begin{cases} 1/f(x) & \text{if} \quad f(x) \neq 0 \\ 1 & \text{if} \quad f(x) = 0 \end{cases} \tag{8}$$

is the inverse element of f under multiplication. This affirmation can be shown calculating

$$\Big((f)\,(1/f) \Big)\,(x) = 1 \quad \forall x \in P \tag{9}$$

where

$$P = \{x : f(x) \neq 0\} \in U \tag{10}$$

If the set P shown above is not an element of U then f is in the equivalence class of zero. Therefore f is in the equivalence class of 1.

The usual definition of an order relation in HR is

$$f < g \text{ iff } \{x \in I : f(x) < g(x)\} \in U$$

With this relation, HR is a totally ordered field.

We have two dual constructions:

- a function from $I = [0, 1]$ to R is a member of the hyper-real numbers field (with an appropriate equivalence relation).
- an application from R to $I = [0, 1]$ is a fuzzy set.

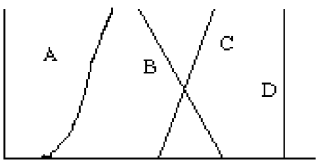

Figure 1: Hyper-real numbers

Fig. 1 shows four different elements of HR, named A, B, C and D. In all the graphics we use, we represent the set I vertically and the set R horizontally. The hyper-real number D, corresponding to the function

$$f_D(x) = d \quad \forall x \in I,$$

can be interpreted as the representation of the real number d.

From the definition of '<', we obtain the following relations: $A < B$, $A < C$, $A < D$, $B < D$ and $C < D$. The comparison between B and C is not trivial, depending on the considered ultrafilter U. With one ultrafilters we can obtain $B < C$ and with another $C < B$.

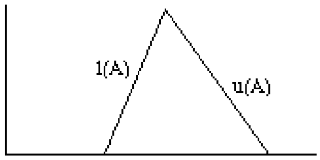

Figure 2: Triangular fuzzy number.

Like we can not find an analytic definition of U, the total ordering defined above has a limited use.

We propose to define a fuzzy relation R to compare hyper-real numbers as follows:

$$R(B, C) = \mu\Big(\{x \in [0, 1] : B(x) < C(x)\} \Big) \tag{11}$$

for any measure μ defined in $[0, 1]$. This fuzzy relation can be interpreted as the proportion of ultrafilters in a family verifying $B < C$.

3 A Nonstandard interpretation of fuzzy numbers

In the graphic representation shown above, we can see how hyper-real numbers and fuzzy numbers have a dual nature. A fuzzy number we call in this paper as follows

Definition A fuzzy number A is defined as a fuzzy set on the space of real numbers R, whose membership function μ_A satisfies the following conditions:

1. μ_A is a mapping from R to the closed interval $[0, 1]$,
2. there exists a unique real number m such that
   (a) $\mu_A(m) = 1$
   (b) μ_A is nondecreasing on $(-\infty, m]$,
   (c) μ_A is nonincreasing on $[m, +\infty)$,

In the above definition we should notice that no continuity is assumed to the membership function. We denote the set of all fuzzy numbers by FR.

Figure 2 shows a classic representation of a triangular fuzzy number. We can see that fuzzy number A is limited by two functions: the lower bound $l(A)$ and the upper bound $u(A)$. Every α-level cut can be expressed as $(l(A)(\alpha), u(A)(\alpha))$. But the functions $l(A)$ and $u(A)$ can also be interpreted as hyper-real numbers, verifying $l(A) < u(A)$.

Analyzing this fact, we interpret a fuzzy number as an interval in the set of hyper-real numbers: The set of

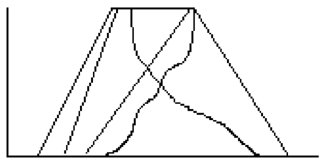

Figure 3: Trapezoid fuzzy number

all hyper-real numbers h verifying $l(A) < h < u(A)$. Figure 3 shows this idea using a trapezoid fuzzy number.

Although we can consider every fuzzy set as an interval of hyper-real numbers, the opposite implication is not true. If we consider two hyper-real numbers f and g verifying $f(0) = g(0)$, $f(1) = g(1)$ and $f(x) < g(x) \forall x \in (0, 1)$, obviously $f < g$ and they are the bounds of an interval of hyper-real numbers, but they cannot be interpreted as the lower bound and the upper bound of a fuzzy set.

The addition and multiplication of fuzzy numbers is deduced from interval arithmetic. Assume that A and B are two fuzzy numbers, which bounds are $(l(A), u(A))$ and $(l(B), u(B))$ respectively. The addition of A and B is the fuzzy number C which bounds are $(l(A) + l(B), u(A) + u(B))$.

The lower bound of the resulting fuzzy set C is the sum of the lower bounds of the fuzzy numbers A and B, and the upper bound of the resulting fuzzy set C is the sum of the upper bounds of the fuzzy numbers A and B. Using α-level sets we obtain the same result. The multiplication of A and B is the fuzzy number which bounds are $(l(A)l(B), u(A)u(B))$.

4 Ranking fuzzy numbers

Using this interpretation of fuzzy sets, we will define a fuzzy relation L with the purpose of ranking fuzzy numbers. At first time, it is necessary the definition of a measure of intervals $\mu(A)$. In this paper, we consider the measure of an interval $A = (l(A), u(A))$ given by the area below the two bounded curves. If $A_i \subset A$, a relative measure may be defined as follows:

$$M_A(Ai) = \frac{\mu(Ai)}{\mu(A)} \quad (12)$$

The fuzzy relation we are building must verify the conditions:

$$L(A, B) = 1 \text{ if } \max(A) < \min(B) \quad (13)$$

$$L(A, B) = 0 \text{ if } \min(A) < \max(B) \quad (14)$$

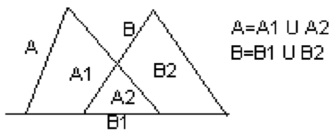

Figure 4: Comparison of fuzzy numbers

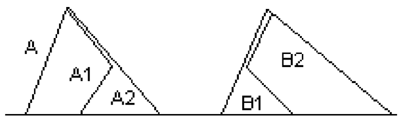

Figure 5: Making subintervals

$$L(A, A) = 1/2 \quad (15)$$

where max and min are interpreted as the maximum and the minimum of all hyper-real numbers in the interval. The last condition is introduced to ensure reciprocity and weak reflexivity.

We considered that several properties related to partitions must be verified by this relation. If it is considered partitions of the sets

$$A = A_1 \cup A_2 \cup \ldots \cup A_n \text{ with } A_i \cap A_j = \emptyset \text{ if } i \neq j$$

$$B = B_1 \cup B_2 \cup \ldots \cup B_m \text{ with } B_i \cap B_j = \emptyset \text{ if } i \neq j$$

The fuzzy relation we are building must verify the next properties:

$$L(A, B) = \sum_{i=1}^{n} M_A(A_i) L(A_i, B) \quad (16)$$

and

$$L(A, B) = \sum_{j=1}^{m} M_B(B_j) L(A, B_j) \quad (17)$$

therefore

$$L(A, B) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_A(A_i) M_B(B_j) L(A_i, B_j) \quad (18)$$

Whith an appropriated selection of the partitions, $L(A_i, B_j)$ can be calculated using the conditions and properties in a great number of situations. The rest of the cases will be discussed bellow. These cases occur when the intersection area and the areas of both sets are the same.

For instance, figure 4 shows two fuzzy numbers with $supp(A) \cap supp(B) \neq \emptyset$, Number $A = (l(A), u(A))$ can be divided in two subintervals A_1 and A_2, and number B can be divided in B_1 and B_2, as shown in figure 5.

We define a new hyperreal number $m(A)$ verifying $m(A)(x) = \min[u(A)(x), l(B)(x)]$.

So $A_1 = (l(A), m(A))$ and $A_2 = (m(A), u(A))$.

By analogy, $m(B)(x) = \max[u(A)(x), l(B)(x)]$, and so $B_1 = (l(B), m(B))$ and $B_2 = (m(B), u(B))$

Like $\max(A_1) < min(B_1)$, and $\max(A_1) < min(B_1)$ then $L(A_1, B_1) = L(A_2, B_2) = 1$. By the way, $\max(A_2) < min(B_2)$, then $L(A_2, B_2) = 1$.

$L(A_2, B_1)$ is the only case unable to calculate applying the conditions (13-18). It happens because the intersection area and the areas of both sets are the same. It can be defined as $L(A_2, B_1) = 1/2$, but we prefer a more elaborated construction. We call (a, h) to the intersection point between $u(a)$ and $l(b)$. If f and g are hyper-real numbers belong to A_2 and B_1 respectively, then we obtain $f(x) < g(x) \forall x \in (h, 1]$ however, if $x \in [0, h)$, it may be happen that $f(x) < g(x)$ or $g(x) < f(x)$.

If we want to calculate $L(A, B)$ and we are in a case unable to calculate applying the conditions (13-18), we define a real function $r_{A,B}[0,1] \longrightarrow [0,1]$ as follows:

$$r_{A,B}(x) = \begin{cases} 1 & \text{if } f(x) < g(x) \begin{cases} \forall f \in A \\ \forall g \in B \end{cases} \\ 0 & \text{if } g(x) < f(x) \begin{cases} \forall f \in A \\ \forall g \in B \end{cases} \\ 1/2 & \text{otherwise} \end{cases} \quad (19)$$

The average of the height of $r_{A,B}$, which is equal to its area, is the value of $L(A, B)$ we define. Thus

$$L(A, B) = \int_0^1 r_{A,B}(x)dx \quad (20)$$

wich is a generalization of (11).

In the example of figures 4 and 5, the function r_{A_2,B_1} is equal to

$$r_{A_2,B_1}(x) = \begin{cases} 1 & \text{if } x \in (h, 1] \\ 1/2 & \text{if } x \in [0, h) \end{cases}$$

and the value of $L(A_2, B_1)$ is $1 - h/2$. So

$$\begin{aligned} L(A, B) &= L(A_1, B_1)M_A(A_1)M_B(B_1) + \\ &+ L(A_1, B_2)M_A(A_1)M_B(B_2) + \\ &+ L(A_2, B_1)M_A(A_2)M_B(B_1) + \\ &+ L(A_2, B_2)M_A(A_2)M_B(B_2) \end{aligned}$$

and

$$L(A, B) = M_A(A_1)M_B(B_1) + M_A(A_1)M_B(B_2) + \\ +(1 - h/2)M_A(A_2)M_B(B_1) + M_A(A_2)M_B(B_2)$$

Therefore

$$L(A, B) = 1 - h/2M_A(A_2)M_B(B_1)$$

It must be noticed that $L(B_1, A_2) = h/2$. Consequently, $L(A, B) + L(B, A) = 1$ which is the reciprocity property.

5 Conclusions

The interpretation of fuzzy numbers as intervals in HR can inspire new definitions of the basic concepts. In this paper, we propose a fuzzy ranking method based on this interpretation which can be used to solve optimization problems.

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References

[1] Drossos, C. Foundations of Fuzzy Sets. A nonstandard approach. *Fuzzy Sets and Systems* 1996.

[2] Furukawa, N. Parametrics Orders on Fuzzy Numbers and their roles in Fuzzy Optimization Problems. *Optimization* 1997.

[3] González, A. Números difusos, Aritmética y comparación. *Algunos Aspectos del Tratamiento de la Información en Inteligencia Artificial. Universidad de Granada* 1991.

[4] Jiménez, M. Ordenación de números borrosos mediante la comparación de sus intervalos esperados. *Actas del Cuarto Congreso de la Asociación Esp de Tecnologías y Lógica Fuzzy* 1994.

[5] Kandel, J. Fuzzy Mathematical Techniques with Applications. *Addison-Wesley* 1986.

[6] Lai, Y. J. y Hwang, C. L. Fuzzy Mathematical Programming, Methods and Applications. *Springer-Verlag* 1992.

[7] Pedrycz, W. Two Nonparametric Models for Fusing Heterogeneous Fuzzy Data. *IEEE Trans. On Fuzzy Systems* Vol. 6, No. 3. 1998.

[8] Robinson, A. Non-Standard Analysis. *Studies in Logic and the Foundations of Mathematics, North-Holland* 1966.

[9] Zadeh, L. A. Similarity Relations and Fuzzy Orderings. *Information Sciences* 3. 1971.