EXCHANGEABILITY OF EXPECTATION, DIFFERENTIAL AND SUPPORT FUNCTION

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Summary

In this work we present sufficient conditions to guarantee the exchangeability among the expected value, the support function and the differential of fuzzy-valued mappings depending on a real-valued parameter.

Keywords: Differentiability, Fuzzy Expected Value, Fuzzy Random Variable, Support Function.

1 INTRODUCTION

There exists a vast literature about the differentiability of fuzzy-valued mappings, see for instance Dubois and Prade [2], Puri and Ralescu [5], Goetschel and Voxman [3] or Kaleva [4]. One of the main problems in working with differentials of fuzzy-valued mappings is that the usual class of fuzzy sets of R^p does not configure a vectorial space with respect to the sum and the product by a scalar induced by Zadeh's Extension Principle [9]. Due to this problem, it becomes sometimes useful to embed that class of fuzzy sets into a normed space, in order to define a classical Fréchet-type differential. This tool has been used in Puri and Ralescu [6] and, recently, in Rojas-Medar *et al.* [8] to embed special classes of fuzzy sets into the space of continuous functions from $[0,1] \times S^{p-1}$ to R.

In order to develop an integration theory for fuzzy-valued random variables, an important question arises in the non-separability of the usual class of fuzzy sets. This problem can be avoided by using the previously referred embedding, because fuzzy-valued mappings become functions taking on values in a compact Banach space.

In some problems in Statistics it is necessary to exchange the expectation of a random variable and the differential with respect to some real-valued parameter. We can combine the above techniques in order to give an answer to this problem of exchangeability in the previous context.

2 PRELIMINARIES

Let $\mathcal{K}_c(\mathrm{R}^p)$ be the class of non-empty compact convex subsets of R^p. The well-known Hausdorff metric on this class is defined by

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

for all $A, B \in \mathcal{K}_c(\mathrm{R}^p)$.

We will denote by $\mathcal{F}_{cc}(\mathrm{R}^p)$ the class of the fuzzy sets $V : \mathrm{R}^p \to [0,1]$ with $V_\alpha \in \mathcal{K}_c(\mathrm{R}^p)$, for all $\alpha \in [0,1]$, and the mapping $\alpha \to V_\alpha$ being continuous with respect to the Hausdorff metric (where the 0-level is intended to be the closed convex hull of the support of V).

On $\mathcal{F}_{cc}(\mathrm{R}^p)$ we can define the metric (Puri and Ralescu [7]) given by

$$d_\infty(A,B) = \sup_{\alpha\in[0,1]} d_H(A_\alpha, B_\alpha)$$

for all $A, B \in \mathcal{F}_{cc}(\mathrm{R}^p)$.

We will denote by $\mathcal{F}_{cl}(\mathrm{R}^p)$ the subclass of fuzzy sets $V \in \mathcal{F}_{cc}(\mathrm{R}^p)$ such that the mapping $\alpha \to V_\alpha$ is Lipschitz.

Puri and Ralescu [6] proved that the metric space $(\mathcal{F}_{cl}(\mathrm{R}^p), d_\infty)$ can be embedded into the space $(C([0,1]\times S^{p-1}), \|\cdot\|_\infty)$ by means of the support function of the fuzzy sets, that is, the mapping

$$j : \mathcal{F}_{cl}(\mathrm{R}^p) \to C([0,1]\times S^{p-1})$$

given by $j(A) = s_A$, for every $A \in \mathcal{F}_{cl}(\mathrm{R}^p)$ (where s_A is the support function of A) is an isometrical embedding. Recently, Rojas-Medar *et al.* [8] have extended this embedding to the class $\mathcal{F}_{cc}(\mathrm{R}^p)$.

Let $(\Omega, \mathcal{A}, P)$ a probability space. A set-valued mapping $X : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$ is said to be a *random set* if it is Borel-measurable when we consider the Hausdorff metric on $\mathcal{K}_c(\mathbb{R}^p)$. A random set is said to be *integrably bounded* if there exists a mapping $h : \Omega \to \mathbb{R}$ with $h \in L^1(\Omega, \mathcal{A}, P)$, such that

$$\sup_{x \in X(\omega)} \|x\| \leq h(\omega) \;\; a.s. \, [P].$$

The expected value of an integrably bounded random set, denoted by $E(X)$, can be defined by means of Aumann's integral [1].

A mapping $X : \Omega \to \mathcal{F}_{cc}(\mathbb{R}^p)$ is said to be a *fuzzy random variable* if it is Borel-measurable when we consider the d_∞ metric on $\mathcal{F}_{cc}(\mathbb{R}^p)$. A fuzzy random variable is said to be *integrably bounded* if the mapping $X_0 : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$, with $X_0(\omega) = (X(\omega))_0$ for all $\omega \in \Omega$ is an integrably bounded random set.

Puri and Ralescu [7] established the concept of expected value for these integrably bounded fuzzy random variables, as the unique fuzzy set such that $(E(X))_\alpha = E(X_\alpha)$, for every $\alpha \in [0, 1]$.

3 MAIN RESULT

Here, we prove our main result, an theorem about the exchanging of differential and expectation for fuzzy random variables. The result is obtained after embedding the fuzzy random variable by means of its support function, so an exchange result with support function is also obtained.

THEOREM 3.1 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $T \subseteq \mathbb{R}^k$ be a non-empty, convex and open subset. Let $X : \Omega \times T \to \mathcal{F}_{cc}(\mathbb{R}^p)$ be a mapping satisfying that*

i) for all $t \in T$, the mapping $X_t : \Omega \to \mathcal{F}_{cc}(\mathbb{R}^p)$ with $X_t(\omega) = X(\omega, t)$, is an integrably bounded fuzzy random variable;

ii) the support function associated with the mapping $X_\omega : T \to \mathcal{F}_{cc}(\mathbb{R}^p)$, where $X_\omega(t) = X(\omega, t)$, is Fréchet differentiable at a neighborhood N of $t_0 \in T$, a.s. $[P]$, for every $(\alpha, r) \in [0,1] \times S^{p-1}$;

iii) there exists a real-valued function $h : \Omega \to \mathbb{R}$, with $h \in \mathrm{L}^1(\Omega, \mathcal{A}, \mathrm{P})$, such that

$$\|\frac{\partial}{\partial t} s_{X(\omega,t)}(\alpha, r)\| \leq h(\omega) \; a.s. \, [P],$$

for all $t \in N$, and for every $(\alpha, r) \in [0,1] \times S^{p-1}$.

Then, the following equalities hold:

$$\frac{\partial}{\partial t} s_{EX(\omega,t_0)}(\alpha, r) = \frac{\partial}{\partial t} E(s_{X(\omega,t_0)}(\alpha, r)) \qquad (1)$$

$$\frac{\partial}{\partial t} E(s_{X(\omega,t_0)}(\alpha, r)) = E\left(\frac{\partial}{\partial t} s_{X(\omega,t_0)}(\alpha, r)\right) \qquad (2)$$

for all $(\alpha, r) \in [0,1] \times S^{p-1}$.

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