THE CARDINALITY OF A FUZZY SUBSET AS A PART OF THE CARDINALITY OF THE WHOLE FUZZY SET

Pedro Canet Estelrich
Departament de Matemàtiques i Informàtica
Universitat de les Illes Balears
Crra.Valldemossa Km. 7'5
07071 Palma de Mallorca, Spain
Email:pcedmi0@ps.uib.es

Jaume Casasnovas Casasnovas
Departament de Matemàtiques i Informàtica
Universitat de les Illes Balears
Crra.Valldemossa Km. 7'5
Email: jccdmi0@ps.uib.es
07071 Palma de Mallorca, Spain

Abstract

In this paper we propose a definition of the cardinal of a finite fuzzy subset in an analogous situation to the crisp one where we know the relationship between the quantities of elements belonging to the subset and to the whole set.
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1. INTRODUCTION:

There are solid motivations for undertaking the problem of cardinality of fuzzy sets. On the one hand, we mean a generalisation of the classical cardinality theory for sets. On the other hand, let us mention applications to areas lying far beyond the pure mathematics, such as the meaning of imprecise quantifiers in natural language statements, aggregations of imprecise predicates, decision-making in a fuzzy environment. For instance, we mean the problem of satisfactory answers in databases to queries of the form "How many x's are **p**". Moreover, we need an ordering relation and arithmetic enabling us to compare cardinalities of fuzzy sets and to calculate with them in order to answer to queries of the form: Are there more x's which are p than x's which are q?
In the subject literature, several approaches to cardinality of fuzzy sets can be found (see Wygralak [4,5], Zadeh[6])
For finite fuzzy sets, in [5] we can found the definitions and the properties of the cardinalities as generalised natural numbers, even their arithmetic and comparisons
in the same sense which we will use in this paper, where we work cardinality in such form that it allows us to quantify concepts as "the subset B have a half elements of A", which appears in natural language where is assumed that a "twenty per cent of young people are very tall"

2. PRELIMINARY REMARKS:

Each fuzzy set in a universe **M** is characterised by a function $A: \mathbf{M} \to [0\ 1]$, which is called the membership function of that fuzzy set . As usual, the set "suppA" = $=\{x\in \mathbf{M}: A(x) \neq 0\}$. The fuzzy sets refered in this paper are *finite* , i.e. "supp A" is a finite crisp set. If C is a crisp finite set, then |C| denote the "cardinal" of C, the "number of elements which belong to C". $B \subset A$ iff $B(x) \leq A(x)$, $\forall x \in \mathbf{M}$.

Besides, we will use the following definitions:

$$A_t = \{x \in \mathbf{M}: A(x) \geq t\} \ ;\ t \in [0\ 1] \quad (1)$$

$$[A]_i = \sup \{t\in [0\ 1]: |A_t| \geq i\} \ ;\ i\in N=\{0,1,2,\dots\} \quad (2)$$

$$m = |A_1| \ ;\ n = |\text{supp } A| \quad (3)$$

Lemma 1: Let A, B finite fuzzy set. The following properties hold true:
a) $[A]_i$ is nonincreasing with respect to i
b) $[A]_i = 1$ for each $i \leq m$
c) $[A]_i = 0$ for each $i > n$
d) $0< [A]_i <1$ for each $m < i \leq n$
e) If $m < i \leq n$, then $[A]_i$ is the ith element of the nonincreasingly ordered sequence of all positive values A(x), including their possible repetitions
f) $B \subset A$ implies $[B]_i \leq [A]_i$ for each $i \in N$

Definition 1: Let A be a fuzzy set, the "cardinality" of A is defined (Wygralak [4,5]) as a fuzzy set |A| in N as follows:

$$|A|(i) = \min \{[A]_i\ ,\ 1 - [A]_{i+1}\} \quad (4)$$

Let us define: $z_A = \min\{i \in N: [A]_i + [A]_{i+1} \leq 1\}$ (5)

Remark 1: By definition:
$|A|(i) = 1 - [A]_{i+1}$ if $i < z_A$, else $[A]_i$

Remark 2 :

a) $[A]_i = 1 - |A|(i - 1)$, for $i \leq z_A$
b) $[A]_i = |A|(i)$, for $i \geq z_A$

Proof: immediate from the definitions

Remark 3 : as a consequence of the above remark :

$$1 - |A|(z_A - 1) = |A|(z_A)$$

Lemma 2 :

a) $|A|(i)$ is no decreasing , for i= 0,1,…z_A-1
b) $|A|(i)$ is nonicreasing for i= z_A, z_A+1,…n
c) if $i < z_A$,then $[A]_i > 0'5$ and $[A]_i \leq 0'5$ if $i > z_A$
d) if $i < z_A - 1$, then $|A|(i) < 0'5$, and, if $i > z_A$, then $|A|(i) \leq 0'5$
e) if $|A|(z_A) < 0'5$ then: $\max\{|A|(i), i=0,1\ldots n\} = |A|(z_A - 1) \geq 0'5$
f) if $|A|(z_A) > 0'5$, then: $\max\{|A|(i), i=0,1\ldots n\} = |A|(z_A) > 0'5$
g) if $|A|(z_A) = 0'5$, then: $\max\{|A|(i), i=0,1\ldots n\} = |A|(z_A) = |A|(z_A - 1) = 0'5$

Lemma 3 : let A,B be finite fuzzy sets. The following two conditions are equivalent:
a) $\forall i \in N: \min\{[A]_i, 1 - [A]_{i+1}\} = \min\{[B]_i, 1 - [B]_{i+1}\}$
b) $\forall i \in N: [A]_i = [B]_i$

Definition 2 : Equipotent fuzzy sets

$$A \sim B \text{ iff } |A|(i) = |B|(i)\ \forall i \in N \quad (6)$$

$$A \sim^s B \text{ iff } |A|(i) - |B|(i) \leq s\ \forall i \in N \quad (7)$$

Theorem 1 (Wygralak[5]) . Let A,B finite fuzzy sets. The condition A~B is equivalent to each of the following conditions:
a) $\forall i \in N: [A]_i = [B]_i$
b) $\forall t \in (0\ 1] : |A_t| = |B_t|$

Remark 4 : If $B \subset A$, then the inequality :

$$|A|(i) \leq |B|(i)\ \forall i \in N$$

does not fulfil, but:
a) $|A|(i) \leq |B|(i)$ if $i < \min\{z_B, z_A\}$
b) $|B|(i) \leq |A|(i)$) if $i > \max\{z_B, z_A\}$

3. THE CARDINALITY OF A FUZZY SUBSET AS A PART OF THE THE CARDINALITY OF THE WHOLE FUZZY SET

Definition 3 : Let A be a fuzzy set in **M**. If $0 < p \leq 1$ is a rational number such that $1/p \in N$ and B is a fuzzy subset of A, then:

$$|B| = p|A| \text{ iff : } |B|(i) = |A|(i/p) \quad \text{for each } i \in N \quad (8)$$

For instance: for p = 1/2

i	$[A]_i$	$\|A\|(i)$	$\|B\|(i)$	$[B]_i$
0	1	0	0	1
1	1	0’1	0’2	1
2	0’9	0’2	0’4	0’8
3	0’8	0’2	0’6	0’6
4	0’8	0’4	0’3	0’3
5	0’6	0’4	0’2	0’2
6	0‘6	0‘6	0	0
7	0’4	0’4	0	0
8	0’3	0‘3	0	0
9	0’3	0’3	0	0
10	0’2	0’2	0	0
11	0’1	0’1	0	0
12	0	0	0	0

where p= ½ ; z_A= 6 ; z_B = 3, |suppA|=11 , |suppB|=5

Remark 5 : From a fuzzy set , A, with cardinality $|A|(j)$, j = 0,1,...n, the cardinality of the fuzzy set, B, $|B|(i) = |A|(i/p)$ results determined with the values of $|A|(j)$, j = 0, 1/p, 2/p, ... (0,2,4,6,… in the above example) and this determination does not change depending of the values of the cardinality $|A|(j)$, $i/p < j < (i+1)/p$ (1,3,5,6,7…in the above example). Moreover, the values of $[B]_i$, i = 0,1,2,… are determined by the values of $|B|(i)$.

Remark 6 : The following situation is possible:

for p = 1/2

i	$[A]_i$	$\|A\|(i)$	$\|B\|(i)$	$[B]_i$
0	1	0	0	1
1	1	0’1	0’2	1
2	0’9	0’2	0’3	0’8
3	0’8	0’2	0’6	0’7
4	0’8	0’3	0’3	0’3
5	0’7	0’4	0’2	0’2
6	0‘6	0‘6	0	0
7	0’4	0’4	0	0
8	0’3	0‘3	0	0
9	0’3	0’3	0	0
10	0’2	0’2	0	0
11	0’1	0’1	0	0
12	0	0	0	0

where : z_A= 6; $|A|(6) = |B|(3) = 0'6$;
$|A|(4) = |B|(2) = 0'3$;
z_B = 3, but $[B]_3 \neq |B|(3)$, in contradiction with Remark 2

Proposition 1: If $z_B/p = z_A$, it is possible to define exactly the cardinality of B from the cardinality of A , iff

$$[A]_{(zB-1)/p+1} = [A]_{zB/p},$$

Proof:
$[A]_{zB/p} = |A|(z_B/p) = |B|(z_B) = 1 - |B|(z_B - 1) =$
$= 1 - |A|((z_B - 1)/p) = [A]_{(zB-1)/p+1}$.

Remark 7: if the condition of the remark 6 does not fulfil, then:

$A \sim^s B$, with $s = [A]_{zA-1/p+1} - [A]_{zA}$

Proposition 2: If $|B| = p|A|$ and $i_M = \max \{i : i/p \le z_A\}$, then:

$$i_M \le z_B \le i_M + 2$$

Proof:

1. Let suppose that $z_B < i_M$, then we have:

$z_B/p < z_A$; $z_B/p + 1 < (z_B + 1)/p \le z_A$;
$(z_B-1)/p < z_A$; $z_B/p + 1 \le z_A-1$

a)
$|A|((z_B-1)/p) \le |A|(z_B/p)$, by lemma 2.a,
So, $|B|(z_B-1) \le |B|(z_B)$
And, $1 - [B]_{zB} \le [B]_{zB}$, and $[B]_{zB} \ge 0'5$, by lemma2.e,f,g

b)
$|B|(z_B) \ge |B|(z_B+1)$, by lemma 2.b
So, $|A|(z_B/p) \ge |A|((z_B+1)/p)$

c)
$(z_B + 1)/p < z_{A,}$ because if $(z_B+1)/p = z_A$, then :
$|A|(z_{B/p}) = 1-[A]_{zB/p+1}$,
$|A|((z_B+1)/p) = |A|(z_A) = [A]_{zA} = [A]_{(zB+1)/p}$
and : $1-[A]_{zB/p+1} \ge [A]_{(zB+1)/p}$, by the above b),
So:
$[A]_{za-1} + [A]_{zA} \le [A]_{zB/p+1} + [A]_{(zB+1)/p} \le$
$\le [A]_{zB/p+1} + 1-[A]_{zB/p+1} = 1$
in contradiction with the definition of z_A (Def.1)

So, $(z_B+1)/p < z_A$, and by lemma 2.a:
$|A|(z_B/p) \le |A|((z_B+1)/p)$

c) from b) and c), it results:
$|A|(z_B/p) = |A|((z_B+1)/p)$, and so:
$|B|(z_B) = |B|(z_B+1)$
but $|B|(z_B+1) \le 0'5$, by lemma 2.c.
So: $|B|(z_B) = [B]_{zB} \le 0'5$

e) from a) and d) , it results :
$[B]_{zB} = 0'5$
and $[B]_{zB+1} = [B]_{zB} = |B|(z_B+1) = |A|((z_B+1)/p = 0'5$
and : $|A|(z_B/p) = |B|(z_B) = 0'5$

$1- [A]_{zB/p+1} = 1- [A]_{(zB+1)/p+1} = 0'5$
but $z_B/p + 1 \le z_A-1$, and $(z_B+1)/p < z_A$ by c)
<u>in contradiction with de definition of z_A</u>

2. Let now suppose that : $z_B > i_M + 2$, then:
$z_B/p > (z_B-1)/p > (z_B-2)/p > i_M/p > z_A$

$|B|(z_B) = |A|(z_B/p) = [A]_{zB/p} \le [A]_{(zB-1)/p} = |A|((z_B-1)/p) =$
$=|B|(z_B-1)$.
Analogously : $[A]_{(zB-1)/p} = |A|((z_B-1)/p)_p \le [A]_{(zB-2)/p} =$
$=|A|((z_B-2)/p) = =|B|(z_B-2)$,
and: $|B|(z_B) \le |B|(z_B-1) \le |B|(z_B-2)$

From where : $1 -[B]_{zB-1} \ge 1-[B]_{zB} \ge [B]_{zB}$
But : $[B]_{zB-1} \ge [B]_{zB}$, by lemma 1.a, and we can assume:
$[B]_{zB-1} = [B]_{zB}$
On the other hand $[B]_{zB-1} > 0'5$, by lemma 2.c, and we also can assume $[B]_{zB} > 0'5$ and $1-[B]_{zB} < [B]_{zB,}$ which results contradictory

Remark 8 : it is possible that $z_B = i_M, i_M+1$ or i_M+2
For instance:

i	[B](i)	|B|(i)	|A|(i)	[A](i)
0	1	0'1	0'1	1
1	0'9	0'3	0'2	0'9
2	0'7	0'5	0'3	0'8
3	0'5	0'5	0'5	0'7
4	0'4	0'4	0'5	0'5
5	0'3	0'3	0'5	0'5
6	0	0	0'5	0'5
7	0	0	0'45	0'45
8	0	0	0'4	0'4
9	0	0	0'35	0'35
10	0	0	0'3	0'3
11	0	0	0'2	0'2
12	0	0	0	0

where p= ½ ; $z_A= 4$; $i_M= 2$; $z_B = 3$, |suppA|=11 , |suppB|=5
and even it is possible to find examples where $z_B = i_M + 2$ and $pi_M + 1 < z_{A,}$, for instance:

i	[B](i)	|B|(i)	|A|(i)	[A](i)
0	1	0'1	0'1	1
1	0'9	0'4	0'2	0'9
2	0'6	0'5	0'3	0'8
3	0'5	0'5	0'4	0'7
4	0'4	0'4	0'5	0'6
5	0'3	0'3	0'5	0'5
6	0	0	0'5	0'5
7	0	0	0'5	0'5
8	0	0	0'5	0'5
9	0	0	0'5	0'5
10	0	0	0'4	0'4
11	0	0	0'4	0'4
12	0	0	0'4	0'4
13	0	0	0'3	0'3
14	0	0	0'3	0'3
15	0	0	0'3	0'3
16	0	0	0	0

where p = 1/3 ; $z_A= 5$; $i_M= 1$; $z_B = 3$, |suppA|=15, |suppB|=5

Proposition 3: If $|B| = p|A|$,

a) $[B]_{i+1} = [A]_{i/p+1}$ for each $i \in N$, such that : $i < i_M$

b) $[B]_i = [A]_{i/p}$, for each $i \in N$, such that : $i > z_B$

Proof:

a) if $i < i_M$, then $i < z_B$, by Prop.2, and $i/p < z_A$, by the definition of i_M.

Therefore: $1 - [B]_{i+1} = |B|(i) = |A|(i/p) = 1 - [A]_{i/p+1}$

b) if $i > z_B$, then $i > i_B$, and $i/p > z_A$

Therefore : $[B]_i = |B|(i) = |A|(i/p) = [A]_{i/p}$

Remark 9 : In order to investigate the relationship between $|A_t|$ and $|B_t|$, for each $t \in [0\ 1]$, we mean that, after the definitions of the Preliminary Remarks, is easy to see $|A_t|=\max\{j : [A]_j \geq t\}$. So, the relationship between $|A_t|$ and $|B_t|$ is a consequence of the relationship between $[B]_i$ and $[A]_{i/p}$, i= 0,1,2,…, which has been proved in the above proposition.

We cannot establish an only relation for each $t \in [0\ 1]$, but it depends of the value of t.

So, we can prove propositions as the following one:

Proposition 4: If $|B| = p|A|$ and $|A_t| \leq z_A$ (which occurs when t > 0'5, by lemma 2.c), then:

$$|A_t| - 1/p|B_t| \leq 1$$

Proof:

For a major simplicity , let $|A_t|=\max\{j : [A]_j \geq t\}= j_M$

If $j_M \leq z_A$, then $j_M - 1 < z_A$;

If $k_M = \max\{k \in N: i/p \leq (|A_t|-1)\}=$

$= \max\{i \in N: i/p \leq (j_M -1)\}$, then: $k_M < i_M$ (defined in Prop.2), and $k_M/p + 1 \leq j_M - 1 + 1 = j_M$

So, by Prop.3.a: $[B]_{kM+1} = [A]_{kM/p+1}$

And $[A]_{kM/p+1} \geq [A]_{jM} \geq t$, by lemma 1.a

but $[B]_{kM+2} = [A]_{(kM+1)/p+1}$ or $[B]_{kM+2} = [A]_{(kM+2)/p}$ and in every case $[B]_{kM+2} < t$, because $(k_M+1)/p + 1 > j_M - 1 + 1 = j_M$, and $(k_M + 2)/p = (k_M + 1)/p + 1/p > j_M - 1 + 1 = j_M$

So, $|B_t|=\max\{i : [B]_i \geq t\}= k_M+1$, and we can write:

$1/p \geq (j_M-1) - k_M/p = (j_M-1) - 1/p(k_M+1-1) =$

$= (|A_t| - 1) - 1/p(|B_t| -1) \leq 1/p$

and operating : $|A_t| - 1/p|B_t| \leq 1$

Remark 10: Analogously, if $|A_t| > z_A$ (which occurs when $t \leq 0'5$, lemma 2.c) , $|A_t| - 1/p|B_t| \leq 1/p$

Proposition 5 : If $|B| = p|A|$:

$$|SuppA| - 1/p|SuppB| < 1/p$$

Proof:

Like A and B are finite fuzzy sets, there are:

$|suppB| = \max\{i : [B]_i > 0\}= i_m \geq z_B$

$|suppA| = \max\{j : [A]_j > 0\}= j_m \geq z_A$

but: $[B]_{im} = |B|(i_m) = |A|(i_m/p) = [A]_{im/p}$

and $j_m - i_m/p < 1/p$, because if $j_m - i_m/p \geq 1/p$, we would have: $j_m \geq (i_m/p +1)/p$, and:

$[B]_{im+1} = |B|(i_m+1) = |A|((i_m+1)/p) = [A]_{(im+1)/p} \geq [A]_j > 0$

which contradicts the definition of i_m .

5. CONCLUSIONS:

The proposed definition of the cardinal of a fuzzy subset allows us to work with the concept of a rational part of the whole set or universe. The relationship between the corresponding values that characterize the cardinal of a fuzzy set in (Wygralak [4]) , has been investigated.

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