ON THE SYMMETRY OF FUZZY SETS

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Summary

In this paper a notion of symmetry of a fuzzy set with a finite support is introduced which takes into account both the intralevel symmetry and the interlevel symmetry. It is shown that the ranked distribution of the relative level cardinalities of a set with the minimal index of symmetry is close to a Pareto distribution. An estimation of the maximal number of fuzzy elements of a fuzzy set with the minimal index of symmetry is given.

Keywords: Fuzzy set, Permutation, Symmetric group, Pareto distribution .

1 INTRODUCTION

In [2] Yu. Shreider gave a philosophical justification of the principle of the minimal symmetry. According to this principle, the minimal symmetry (in a broad sense) is characteristic for natural systems. Applying the principle of the minimal symmetry to fuzzy sets that can be considered as natural (by their origin) we may draw conclusions concerning their membership functions. The concept of symmetry introduced in this paper being combined with the principle of the minimal symmetry allows to give an explanation of the appearance of Pareto distributions in some "fuzzy" situations (for example of the effect of "gross tails" in financial modeling, see [1, chapter 4]).

2 THE INDEX OF SYMMETRY OF A FUZZY SET

Let us consider a fuzzy set with a finite support X and a membership function μ. Let μ_1, μ_2, ..., μ_k be the elements of $\mu(X)$ and let

$$X_i = \{x \mid \mu(x) = \mu_i\},$$

$n_i = |X_i|$, and $n = |X|$ ($| \cdot |$ stands for the number of elements). Suppose that the sequence n_i is nonincreasing. A permutation g of X is compatible with the membership function μ iff all sets X_i are invariant with respect to g. So the following group

$$S(X,\mu) = \prod_{i=1}^{k} S(X_i)$$

can be treated as the symmetric group of the fuzzy set (X,μ) (given a set A, by $S(A)$ we denote the symmetric group of A). Clearly

$$|S(X,\mu)| = \prod_{i=1}^{k} n_i!.$$

The group $S(X,\mu)$ takes into account only one type of the symmetry of (X,μ) (its "intralevel" symmetry). We are going to construct the group of the "interlevel" symmetry.

Let X' be a subset of X. We say that $V \subset X'$ is a colevel in X' if V contains exactly one element of each level of X'. In other words the restriction $\mu : V \to \mu(X')$ is one-to-one. By a colevel decomposition of (X,μ) we understand a representation

$$X = \bigcup_{j=1}^{l} V_j \tag{1}$$

such that V_t is a colevel in

$$\bigcup_{j=t}^{l} V_j$$

for every $t = 1,\ldots,l$. We put

$$m_j = |V_j|$$

for $j = 1, \ldots, l$. It can be easily seen that $l = n_1$ and $k = m_1$.

Given a colevel decomposition (1), let $S'(X, \mu)$ be the group of permutations of X which leave every colevel from the decomposition invariant. It is clear that $S'(X, \mu)$ is determined uniquely up to isomorphism. We have

$$S'(X, \mu) = \prod_{j=1}^{l} S(V_j)$$

and

$$|S'(X, \mu)| = \prod_{j=1}^{l} m_j!.$$

The group

$$S(X, \mu) \times S'(X, \mu)$$

seems to be more suitable to evaluate the total symmetry of (X, μ) than the "traditional" symmetric group $S(X, \mu)$.

We say that the number

$$s = \frac{1}{n!} \times \prod_{i=1}^{k} n_i! \times \prod_{j=1}^{l} m_j!\,.$$

is the *index of symmetry* of (X, μ).

It can be easily seen that $s \leq 1$. Indeed, let us consider the map

$$S(X, \mu) \times S'(X, \mu) \to S(X, \mu)$$

defined by $(a, a') \mapsto a' \circ a$ and show that it is injective. Let $(a, a') \neq (b, b')$. First suppose that $a \neq b$. Then $a(x) \neq b(x)$ for some $x \in X$. So $a(x)$ and $b(x)$ belong to different colevels, and $a'(a(x)) \neq b'(b(x))$ because a' and b' preserve colevels. If $a = b$ but $a' \neq b'$ then there exists $y \in X$ such that $a'(y) \neq b'(y)$. So, for $x = a^{-1}(y)$ we have $a'(a(x)) \neq b'(b(x))$.

We have $s = 1$ if either $k = 1$ or $k = n$. In the following section we discuss situations when s takes its minimal value.

3 FUZZY SETS WITH THE MINIMAL INDEX OF SYMMETRY

To describe fuzzy sets (X, μ) with the minimal value of the index of symmetry we follow ideas from [2].

We say that a pair (i, j) is a Pareto point (with respect to a colevel decomposition (1) of (X, μ)) if

$$X_i \cap V_j \neq \emptyset,$$

$$X_i \cap V_{j+1} = \emptyset, \qquad X_{i+1} \cap V_j = \emptyset.$$

Theorem 1 *Let (X, μ) have the minimal index of symmetry s (with n and $k \geq 4$ fixed). Then the sequence of all Pareto points can be divided into three nonempty parts:*

$$(1, n_1), \quad (2, n_2) \quad , \ldots, (t-1, n_{t-1}), \tag{2}$$

$$(t, n_t) = (m_p, p), \tag{3}$$

$$(m_{p-1}, p-1), \quad (m_{p-2}, p-2) \quad , \ldots, (m_1, 1) \tag{4}$$

such that the following conditions are verified. For Pareto points (i, j), $(i-1, j')$ from (2) we have $n_{i-1} - n_i \geq 2$ and

$$n_{i-1} m_{j'} - m_j \leq n_i m_j \leq n_{i-1} m_{j'} + m_{j'}$$

Analogous inequalities hold for Pareto points from (4).

Example 1 *Let $n = 10$. We have the minimal index of symmetry $s \approx 0.02$ in the following cases:*

(1) $n_1 = 4$, $n_2 = 2$, $n_3 = n_4 = n_5 = n_6 = 1$;

(2) $n_1 = 4$, $n_2 = n_3 = 2$, $n_4 = n_5 = 1$;

(3) $n_1 = 6$, $n_2 = 2$, $n_3 = n_4 = 1$ (the dual of (1));

(4) $n_1 = 5$, $n_2 = 3$, $n_3 = n_4 = 1$ (the dual of (2));

Theorem 2 *The number of levels k such that*

$$k \ln k \approx n$$

provides the minimal value of the index of symmetry.

¿From Theorem 1 it follows that Pareto points are close to a hyperbola $nm = const$ and the distribution of n_i/n can be considered as a discrete Pareto distribution. The proof of Theorem 1 is rather technical. We give its "approximate" version.

Let $y = f(x)$ be a smooth decreasing function approximating $m_j = f(n_i)$ at Pareto points. Using the Stirling formula we present $\ln(n! \cdot s)$ as the sum of the following summands:

$$\sum_{i=1}^{k} (0.5 \ln 2\pi + 0.5 \ln n_i + n_i \ln n_i - n_i)$$

and

$$\sum_{j=1}^{l} (0.5 \ln 2\pi + 0.5 \ln m_j + m_j \ln m_j - m_j).$$

Then, up to a constant (depending on k and n), $\ln s$ is approximately equal to the sum of the following two integrals:

$$\int_1^k (0.5 \ln y + y \ln y)\, dx$$

and

$$\int_1^l (0.5\ln 2\pi + 0.5\ln x + x\ln x)\,dy. \tag{5}$$

The integral in (5) can be presented as

$$-\int_1^k (0.5\ln 2\pi + 0.5\ln x + x\ln x)y'\,dx.$$

So we have to minimize

$$\int_1^k \Phi\,dx$$

where

$$\Phi = (0.5+y)\ln y - 0.5\ln 2\pi \cdot y' - (0.5+x)\ln x \cdot y'$$

under the following condition

$$\int_1^k y\,dx = n.$$

We put

$$F(x,y,y') = \Phi + \lambda y.$$

Now applying the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

we get

$$\frac{1}{2x} + \ln x + \frac{1}{2y} + \ln y = const.$$

Therefore

$$xy \cdot \exp\left(\frac{1}{2x}\right) \cdot \exp\left(\frac{1}{2y}\right) = const. \tag{6}$$

Formula (6) shows that if x and y are not too small then $xy \approx const$, i.e. the relation between x and y is similar to that in a Pareto distribution.

Now assume that the segment $[0;1]$ is divided into k equal intervals by the range of the membership function μ. If

$$1 = \mu_1 > \mu_2 > \ldots > \mu_k = \frac{1}{k}$$

then the number of fuzzy elements in X, that is

$$\tilde{n} = \sum_{i=1}^{k} \mu_i n_i,$$

is maximal. The following estimation can be obtained

$$\tilde{n} \approx n + \ln k - k.$$

If $n \approx k\ln k$ (as in Theorem 2) then

$$\tilde{n} \approx n \cdot \left(1 + \frac{1}{k} - \frac{1}{\ln k}\right).$$

For example if $n = 100$ then $k = 30$ and $\tilde{n} \approx 74$.

References

[1] A. Shiryaev (1998). The Fundamentals of Stochastic Financial Mathematics. Phasis, Moscow, 1998, 1018 pp.

[2] Yu. Shreider, A. Sharov (1982). Systems and Models. Radio and Communications, Moscow, 1982, 152 pp.