ON A SIMILARITY RATIO*

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Summary

In [3] an index of similarity among concepts, playing an important role in the Analogical Case-Based Reasoning there shown, is introduced. That index is here studied as a fuzzy relation by analysing for which continuous t-norms T is a T-indistinguishability. That index is also considered from a metric viewpoint.

Furthermore, a series of dues are given with regard to the possible choice of a numerical threshold α best suited for considering two concepts A and B as similar when their index of similarity exceeds the value α.

Key words: Similarity ratio, fuzzy relation, T-indistinguishability.

1. The mathematical study of fuzzy equivalences, or T-indistinguishabilities, has led to the publication of a good many papers (see [9], [14], [5], etc.). It is worthy of note that, within Theoretical Fuzzy Logic, the mathematical analysis of T-indistinguishabilities is a field that is important in itself and of undeniable interest for fuzzy logic applications, such as control theory, for instance (see [6], etc.). Furthermore, there are papers in which given indices are used to measure similarity and where fuzzy logic concepts, that could possibly reinforce the theoretical framework of the proposed methodology, are not taken into account. This is the case of [3], an interesting paper in which what is referred to as the "ratio of similarity" between two concepts A and B, with respective numerical signatures $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ (that can be easily considered as belonging to $[0,1]^n$) related to an n-tuple $P = \{p_1, \ldots, p_n\}$ of relevant properties of the domain in question, plays a decisive role. In the above paper, the **ratio of similarity** is taken to be the number

$$\text{sim}\,(A,B) = \frac{\sum_{i=1}^{n} \text{Min}(a_i, b_i)}{\text{Max}\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right)} \in [0,1], \qquad (1)$$

which defines the crisp relations of analogy given by $A\,\mathbf{sim}_\alpha\, B$ if and only if $\text{sim}(A,B) \geq \alpha$. The function "sim" can evidently be viewed as a fuzzy relation.

2. Let $E = \{x_1, \ldots, x_n\}$ be a finite universe of discourse and $\mathcal{F}(E) = [0,1]^E$ be the set of all the fuzzy subsets of E. The fuzzy relation $\mathbf{sim} : \mathcal{F}(E) \times \mathcal{F}(E) \to [0,1]$, defined by

$$\mathbf{sim}(A,B) = \frac{\sum_{i=1}^{n} \text{Min}(A(x_i), B(x_i))}{\text{Max}\left(\sum_{i=1}^{n} A(x_i), \sum_{i=1}^{n} B(x_i)\right)},$$

is the above-mentioned formula (1), bearing in mind that $A(x_i) = a_i$ and $B(x_i) = b_i$. Obviously, $\mathbf{sim}(A,B) \in [0,1]$ for any A and B of $\mathcal{F}(E)$. Also, $\mathbf{sim}(A,A) = 1$ and $\mathbf{sim}(A,B) = \mathbf{sim}(B,A)$ for any A and B of $\mathcal{F}(E)$; that is, **sim** is a reflexive and symmetrical fuzzy relation in $\mathcal{F}(E)$ (see [7]). Furthermore, it can be proved that it is a separating relation, that is,

Theorem 1 $\mathbf{sim}(A,B) = 1$ if and only if $A = B$.

Note also that $\mathbf{sim}(A,B) = 0$ if and only if $A(x_i) = 0$ or $B(x_i) = 0$ is satisfied for every $i \in \{1, \ldots, n\}$. That is to say

Theorem 2 It is $\mathbf{sim}(A,B) = 0$ if and only if $A \cap B = \varphi_\emptyset$, by defining as usual $(A \cap B)(x_i) = \text{Min}(A(x_i), B(x_i))$ for every $x_i \in E$.

Were **sim** to be a T-indistinguishability for a t-norm T **sim** would have to be T-transitive (see [14]), that is, $T(\mathbf{sim}(A,B), \mathbf{sim}(B,C)) \leq \mathbf{sim}(A,C)$ for any A, B and C of $\mathcal{F}(E)$. Let $\mathcal{F}(\text{Prod})$ be the family of t-norms of the product, that is, the t-norms $T_\varphi = \varphi^{-1} \circ \text{Prod} \circ (\varphi \times \varphi)$, where $\varphi : [0,1] \to [0,1]$ is continuous, strictly increasing and such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Let us see , firstly, that if $T \in \mathcal{F}(\text{Prod})$ ó $T = \text{Min}$, then **sim** is not T-transitive. To this end,

Lemma 3 If **sim** is T-transitive, then $T(a,b) = 0$ for every $(a,b) \in [0,1]^2$ such that $a + b \leq 1$.

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Theorem 4 If $T = \text{Min}$ or $T \in \mathcal{F}(\text{Prod})$, **sim** is not T-indistinguishability.

Proof. Suffice it to take a and b such that $0 < a, b \leq 1/2$, then $a+b \leq 1$ and, nevertheless, $\text{Min}(a,b) > 0$ and $T(a,b) = \varphi^{-1}(\varphi(a)\cdot\varphi(b)) > \varphi^{-1}(0) = 0$. ■ However,

Theorem 5 **sim** is a W-indistinguishability.

Proof: Let A, B and C of $\mathcal{F}(E)$, where $E = \{x_1, \ldots, x_n\}$ and let us denote $a_i = A(x_i)$, $b_i = B(x_i)$ and $c_i = C(x_i)$, for each $i \in \{1, \ldots, n\}$, and $a = \sum_{i=1}^n a_i$, $b = \sum_{i=1}^n b_i$ and $c = \sum_{1=1}^n c_i$. It is a matter of proving that $W(\mathbf{sim}(A,B), \mathbf{sim}(B,C)) \leq \mathbf{sim}(A,C)$ or, alternatively, that

$$\mathbf{sim}(A,B) + \mathbf{sim}(B,C) \leq \mathbf{sim}(A,C) + 1 \qquad (2)$$

Suppose, firstly, that $b \geq \text{Max}(a,c)$; in this case, the inequality to be proven is:

$$\frac{\sum_{i=1}^n \text{Min}(a_i,b_i)}{b} + \frac{\sum_{i=1}^n \text{Min}(b_i,c_i)}{b} \leq \frac{\sum_{i=1}^n \text{Min}(a_i,c_i)}{\text{Max}(a,c)} + 1.$$

As $\text{Min}(a_i,b_i) + \text{Min}(b_i,c_i) \leq a_i + b_i$ and $\text{Min}(a_i,b_i) + \text{Min}(b_i,c_i) \leq b_i + c_i$ for every $i \in \{1, \ldots, n\}$, then $\text{Min}(a_i,b_i) + \text{Min}(b_i,c_i) \leq \text{Min}(a_i,c_i) + b_i$, hence, by summing in i and dividing by b, we get

$$\frac{\sum_{i=1}^n \text{Min}(a_i,b_i)}{b} + \frac{\sum_{i=1}^n \text{Min}(b_i,c_i)}{b} \leq \frac{\sum_{i=1}^n \text{Min}(a_i,c_i)}{b} + 1,$$

and as $\frac{\sum_{i=1}^n \text{Min}(a_i,c_i)}{b} \leq \frac{\sum_{i=1}^n \text{Min}(a_i,c_i)}{\text{Max}(a,c)}$ we get the inequality we sought to prove.

Suppose, secondly, that $a \geq \text{Max}(b,c)$. Consider the following partition of the set of indices:

$$\begin{aligned}
I &= \{i \in \{1,\ldots,n\}; a_i \geq b_i \geq c_i\}\\
J &= \{i \in \{1,\ldots,n\}; a_i > c_i > b_i\}\\
K &= \{i \in \{1,\ldots,n\}; b_i > a_i > c_i\}\\
L &= \{i \in \{1,\ldots,n\}; b_i > c_i \geq a_i\}\\
M &= \{i \in \{1,\ldots,n\}; c_i \geq a_i > b_i\}\\
N &= \{i \in \{1,\ldots,n\}; c_i > b_i \geq a_i\}
\end{aligned}$$

and say $a_I = \sum_{i\in I} a_i$, $b_I = \sum_{i\in I} b_i$, $c_I = \sum_{i\in I} c_i$ and so on for each element of the partition. Using this notation, we get the relations

$$\begin{array}{lll} a_I \geq b_I \geq c_I & b_K > a_K > c_K & c_M \geq a_M > b_M \\ a_J > c_J > b_J & b_L > c_L \geq a_L & c_N > b_N \geq a_N \end{array} \qquad (3)$$

and then $\mathbf{sim}(A,B) + \mathbf{sim}(B,C) = \frac{b_I+b_J+a_K+a_L+b_M+a_N}{a} + \frac{c_I+b_J+c_K+c_L+b_M+b_N}{\text{Max}(b,c)} \leq \frac{b_I+b_J+a_K+a_L+b_M+a_N}{a} + \frac{c_I+b_J+c_K+c_L+b_M+b_N}{b}$.

If the last member of the above expression is found to be less than or equal to

$$\mathbf{sim}(A,C) + 1 = \frac{c_I + c_J + c_K + a_L + a_M + a_N}{a} + 1$$

we will have proven that the inequality (2) holds in the event that $a \geq \text{Max}(b,c)$. However, this is tantamount to proving

$$\frac{b_I + b_J + a_K + b_M}{a} + \frac{c_I + b_J + c_K + c_L + b_M + b_N}{b} \leq \frac{c_I + c_J + c_K + a_M}{a} + 1 \text{ that amounts to}$$

$$\begin{array}{l} b_I b + b_J b + a_K b + b_M b + c_I a + c_K a + c_L a \leq \\ c_I b + c_J b + c_K b + a_M b + b_I a + b_K a + b_L a. \end{array} \qquad (4)$$

From (3) it is

$$b_J b + b_M b + c_L a \leq c_J b + a_M b + b_L a; \qquad (5)$$

also, since $b \leq a$ and $b_I - c_I \geq 0$, it is

$$b_I b + c_I a \leq b_I a + c_I b; \qquad (6)$$

and since $c_K < a_K < b_K$ it is

$$a_K b + c_K a \leq b_K a + c_K b. \qquad (7)$$

By summing (5), (6) and (7) member by member, we get the inequality (4).

Finally, suppose that $c \geq \text{Max}(a,b)$. Then, as in the above case, $\mathbf{sim}(C,B) + \mathbf{sim}(B,A) \leq \mathbf{sim}(C,A) + 1$ and since **sim** is symmetric, (2) again holds; hence, **sim** is W-transitive.■

3. The fuzzy relation **sim** is not only T-transitive for the Łukasiewicz t-norm $T = W$, there also exist t-norms T_φ of the Łukasiewicz family, $\mathcal{F}(W)$ (that is, $T_\varphi = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$ and φ as above when the family of the Product, $\mathcal{F}(\text{Prod})$, was introduced before Lemma 3) for which **sim** is T_φ-transitive. In fact, it is easy to prove

Theorem 6 If $T_\varphi \in \mathcal{F}(W)$ is such that $T_\varphi(x,y) \leq W(x,y)$ for each $(x,y) \in [0,1]^2$, **sim** is a T_φ-indistinguishability.

For example, if $\varphi_n(x) = x^n$, then $T_{\varphi_n} = \varphi_n^{-1} \circ W \circ (\varphi_n \times \varphi_n) \leq W$ in $[0,1]^2$ and then **sim** is T_{φ_n}-transitive and a T_{φ_n}-indistinguishability for any natural number n.

Theorem 7 If $T_\varphi \in \mathcal{F}(W)$ and **sim** is a T_φ-indistinguishability, then $N_{id}(x) \leq N_\varphi(x)$ for every $x \in [0,1]$, where N_{id} and N_φ are the negations associated with the identity and with φ, respectively.

Proof. If an $a \in [0,1]$ existed such that $N_\varphi(a) = \varphi^{-1}(1-\varphi(a)) < N_{id}(a) = 1-a$, then we could consider b such that $N_\varphi(a) < b < 1-a$, hence $\varphi(a) + \varphi(b) >$

$\varphi(a)+\varphi(N_\varphi(a)) = \varphi(a)+\varphi(\varphi^{-1}(1-\varphi(a))) = 1$; and, therefore, $T_\varphi(a,b) = \varphi^{-1}(\varphi(a)+\varphi(b)-1) > \varphi^{-1}(0) = 0$ which contradicts lemma 3 since $a+b<1$. ■ For the characterization of Strong-Negations, see [10].

As an application of the above result, consider that if $\varphi(x) = x^\alpha$, where $\alpha < 1$ is a real number, then **sim** is not T_φ-transitive. Since $x^\alpha \geq x$ if and only if $(1-x)^{1/\alpha} \geq (1-x^\alpha)^{1/\alpha} = N_\varphi(x)$, it is $N_\varphi(x) \leq (1-x)^{1/\alpha} < 1-x = N_{id}(x)$ if $x \in (0,1)$. Therefore, **sim** cannot be T_φ-transitive.

Note: It is easy to prove that for any function φ that satisfies $\varphi(x) \leq x$ for every $x \in [0,1]$, it is $N_{id} \leq N_\varphi$

The reciprocal of last theorem does not hold. Indeed, the function $\varphi : [0,1] \to [0,1]$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \geq 4/5 \\ \frac{5}{4}x^2 & \text{if } 0 \leq x < 4/5 \end{cases}$$

is continuous, strictly increasing and satisfies $\varphi(0) = 0$, $\varphi(1) = 1$; additionally $\varphi(x) \leq x$ holds for every $x \in [0,1]$. Then, taking into account the above observation, is $N_{id} \leq N_\varphi$. Let us see, however, that **sim** is not T_φ-transitive.

Let $E = \{x_1, x_2, x_3\}$ and consider A, B and C, defined as shown by the table below:

A	B	C
1/5	2/5	2/5
2/5	2/5	2/5
2/5	1/5	0
$\sum A(x_i) = 1$	$\sum B(x_i) = 1$	$\sum C(x_i) = 4/5$

Then, as $\mathbf{sim}(A,B) = 4/5$, $\mathbf{sim}(B,C) = 4/5$ and $\mathbf{sim}(A,C) = 3/5$ it is $T_\varphi(\mathbf{sim}(A,B), \mathbf{sim}(B,C))= \varphi^{-1}(\varphi(4/5)+\varphi(4/5)-1) = \varphi^{-1}(3/5) > \mathbf{sim}(A,B)$.

4. Like any fuzzy set, the relation **sim** has its associated α-cuts (see. [7]); that is, the classical relations $\mathbf{sim}_\alpha$ in $\mathcal{F}(E)$, defined by

$$A\, \mathbf{sim}_\alpha\, B \text{ if and only if } \mathbf{sim}(A,B) \geq \alpha.$$

Such relations are referred to in [3] as relations of **analogy** and, obviously, are all reflexive and symmetrical, although the only one that is transitive and, therefore, a relation of equivalence, corresponds to $\alpha = 1$.

Theorem 8 $\mathbf{sim}_\alpha$ is transitive if and only if $\alpha = 1$.

Proof. If $\alpha = 1$, it follows from theorem 1 that $\mathbf{sim}_\alpha$ is the relation of equality and is, therefore, transitive. If $\mathbf{sim}_\alpha$ is transitive and suppose that $\alpha < 1$, consider ε, $0 < \varepsilon < \alpha$, and A, B, $C \in \mathcal{F}(E)$, where $E = \{x_1, x_2, x_3\}$, defined according by the table:

A	B	C
α	α	$\alpha-\varepsilon$
0	$1-(\alpha+\varepsilon)$	0
0	ε	$1-(\alpha-\varepsilon)$
$\sum A(x_i) = \alpha$	$\sum B(x_i) = 1$	$\sum C(x_i) = 1$

Hence $\mathbf{sim}(A,B) = \alpha$, $\mathbf{sim}(B,C) = \alpha$, as $\varepsilon \leq 1-(\alpha-\varepsilon)$ if and only if $\alpha \leq 1$, and $\mathbf{sim}(A,C) = \alpha-\varepsilon < \alpha$; then, $\mathbf{sim}_\alpha$ is not transitive. ■ It is clear, pursuant to theorem 1 that $\mathbf{sim}_1 = \{(A,A); A \in \mathcal{F}(E)\}$, whereas $\mathbf{sim}_0 = \mathcal{F}(E) \times \mathcal{F}(E)$ and, pursuant to theorem 2 it is $\mathbf{sim}_0 - \left(\bigcup_{\alpha>0} \mathbf{sim}_\alpha\right) = \{(A,B) \in \mathcal{F}(E) \times \mathcal{F}(E); A \cap B = \varphi_\emptyset\}$.

5. The fact that the fuzzy relation **sim** is T-transitive for some t-norms $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$ means that the question can be addressed from a metric viewpoint, thanks to the following result

Lemma 9 The necessary and sufficient condition for a fuzzy relation $S : E \times E \to [0,1]$ to be a T-indistinguishability for $T \in \mathcal{F}(W)$ where $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$, is that the function $D : E \times E \to [0,1]$ defined by $D = 1 - \varphi \circ S$, is a distance.

The case $\varphi = \text{id}$ was considered in [14] and the general case in [12]. Consequently,

Theorem 10 If $T \in \mathcal{F}(W)$, with $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$, the necessary and sufficient condition for the fuzzy relation **sim** to be a T-indistinguishability is that the function $D_\varphi = 1 - \varphi \circ \mathbf{sim}$ is a distance.

Moreover, D_φ is "separating", in the sense that $D_\varphi(A,B) = 0$ if and only if $A = B$, follows from theorem 1.

If $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$ is a t-norm of $\mathcal{F}(W)$ for which **sim** is T-transitive, it is obvious that

$$A\, \mathbf{sim}_\alpha\, B \quad \text{if and only if} \quad D_\varphi(A,B) \leq 1 - \varphi(\alpha).$$

holds, and the neighbourhoods

$$E_{1-\varphi(\alpha)}(A) = \{B \in \mathcal{F}(E); D_\varphi(A,B) \leq 1-\varphi(\alpha)\},$$

in the respective metric space $(\mathcal{F}(E), D_\varphi)$ afford the equivalent definitions:

$$A\, \mathbf{sim}_\alpha\, B \ \textbf{ iff } \ B \in E_{1-\varphi(\alpha)}(A) \ \textbf{ iff } \ A \in E_{1-\varphi(\alpha)}(B).$$

That is, two functions A and B are linked by the classical relation $\mathbf{sim}_\alpha$ (are "analogical" with the threshold α, as stated in [3]) if and only if the distance between them is at most $1 - \varphi(\alpha)$. Obviously, it is $E_{1-\varphi(\alpha)}(A) = \{A\}$ if and only if $\alpha = 1$ and, of course, the neighbourhoods $E_{1-\varphi(\alpha)}(A)$ are **clusters** of analogical functions.

Pursuant to the above, if $0 \leq \alpha \leq \beta \leq 1$, then $\mathbf{sim}_1 \subset \mathbf{sim}_\beta \subset \mathbf{sim}_\alpha \subset \mathbf{sim}_0$. That is, if the values of the threshold α in $[0,1]$ are high, each $A \in \mathcal{F}(E)$ is analogical to only a few $B \in \mathcal{F}(E)$, whereas if they are low, it is similar to a lot. Therefore, it is reasonable to think that, as a general rule, the **best** relations of analogy $\mathbf{sim}_\alpha$ will correspond to intermediate values of the threshold α. Furthermore, as we are likely to want to evade logically incompatible pairs (A,B) (that is, such that $A \cap B = \varphi_\emptyset$, see [13]), the value for the threshold should be taken from the intermediate region closest to the value $\alpha = 1$; indeed, the paper referenced as [3] considers the threshold $\alpha = 0.7$ which the authors claim to have attained experimentally.

Besides classical relations $\mathbf{sim}_\alpha$, for $\alpha < 1$ are not transitive, they have a kind of restricted transitivity that can allow some control of the analogical reasoning systems in which they are used. In fact, as it is always

$$\mathbf{sim}(A,C) \geq W(\mathbf{sim}(A,B), \mathbf{sim}(B,C)),$$

inequalities $\mathbf{sim}(A,B) \geq \alpha$ and $\mathbf{sim}(B,C) \geq \alpha$ imply $\mathbf{sim}(A,C) \geq W(\alpha,\alpha)$, that is, a lower bound $W(\alpha,\alpha)$ of $\mathbf{sim}(A,C)$ is obtained. Nevertheless, if it is $W(\alpha,\alpha) = 0$ this inequality is not at all informative on the values of $\mathbf{sim}(A,C)$.

To have $W(\alpha,\alpha) = \mathrm{Max}(0, 2\alpha - 1) > 0$ it suffices to have $\alpha > 0.5$. Consequently, if $\alpha > 0.5$ the scheme:

$$\frac{\begin{array}{c} A \;\; \mathbf{sim}_\alpha \;\; B \\ B \;\; \mathbf{sim}_\alpha \;\; C \end{array}}{A \; \mathbf{sim}_{W(\alpha,\alpha)} \; C,}$$

holds with $W(\alpha,\alpha) > 0$, and the classical relation $\mathbf{sim}_\alpha$ shows this restricted transitivity.

As it was proven beforehand, if $T_\varphi \in \mathcal{F}(W)$ verifies $T_\varphi \leq W$ the fuzzy relation **sim** is T_φ transitive, and then: $\mathbf{sim}(A,C) \geq W(\alpha,\alpha) \geq T_\varphi(\alpha,\alpha)$. When it is also $T_\varphi(\alpha,\alpha) > 0$, a small and then not too good lower bound than $W(\alpha,\alpha)$ is reached for $\mathbf{sim}(A,C)$ if $\alpha > 0.5$. Because of with $\varphi(x) = x^2$ it is $T_\varphi \in \mathcal{F}(W)$ and $T_\varphi \leq W$, the number $\sqrt{\mathrm{Max}(0, 2\alpha^2 - 1)}$ is such a bound provided it is positive, a situation that happens if $\alpha > \sqrt{0.5} \simeq 0.707107$. Then, it is enough to take $\alpha > 0.7072$ to be sure that the relation $\mathbf{sim}_\alpha$ enjoys the restricted transitivity shown by:

$$\frac{\begin{array}{c} A \;\; \mathbf{sim}_\alpha \;\; B \\ B \;\; \mathbf{sim}_\alpha \;\; C \end{array}}{A \; \mathbf{sim}_{T_\varphi(\alpha,\alpha)} \; C,}$$

with $T_\varphi(\alpha,\alpha) = \sqrt{\mathrm{Max}(0, 2\alpha^2 - 1)} > 0$. Perhaps a refinement of the experiments that in [3] conducted to use $\alpha > 0.7$ can show that the ratio **sim** was taken there as T_φ-transitive with $\varphi(x) = x^2$ and not as W-transitive. In that case, A and B can be taken as "analogous" if the "distance" between both is less or equal than $1 - \varphi(\alpha) = 1 - (\sqrt{5})^2 = 0.5$

Each application is likely to call for an experimental determination of the maximum threshold. However, **an upper bound β of the maximum threshold can be determined** for each family $\{A_1, \ldots, A_n\}$ of functions as follows. Calculate $\mathbf{sim}(A_i, A_j) = \beta_{ij}$ for each pair of indices i,j, and let $\beta = \mathrm{Min}\{\beta_{ij}; 1 \leq i,j \leq n\}$; as $A_i \, \mathbf{sim}_\beta \, A_j$ for each pair A_i, A_j, then it is also $A_i \, \mathbf{sim}_\alpha \, A_j$ for any $\alpha \leq \beta$.

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