On Implicative Closure Operators in Approximate Reasoning

R. O. Rodríguez[1], **F. Esteva**[2], **P. Garcia**[2] **and L. Godo**[2]

[1] Dpto. de Computación, Fac. Ciencias Exactas y Naturales
Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
ricardo@dc.uba.ar

[2] Institut d'Investigació en Intel·ligència Artificial, CSIC
Campus Univ. Autònoma Barcelona, 08193 Bellaterra, Spain
{esteva,pere,godo}@iiia.csic.es

Summary

This paper introduces a new definition of fuzzy closure operator called Implicative Closure Operators. The Implicative Closure Operators generalize some notions of fuzzy closure operators given by differents authors. We show that the Implicative Closure Operators capture some usual Consequence Relations used in Approximate Reasoning, like the Approximation and Proximity entailments defined by Dubois et al. [5] and the Natural Inference Operator defined by Boixader and Jacas [1].

Keywords: Implication Measures, Approximate Reasoning, Fuzzy Closure Operators, Closure Systems and Fuzzy Consequence Relations.

1 Introduction

In the last years, many works have been devoted to extend the notions of closure operators, closure systems and consequence relations from two valued logic to many valued logic. The best well-known approach to many-valued closure operators is due to Pavelka [11]. He defines such operators (in the standard sense of Tarski) as mappings from fuzzy sets of formulas to fuzzy sets of formulas, i.e.,

Definition 1 (Fuzzy Closure Operator)
A Fuzzy Closure Operator on a language $\mathcal{L}$ is a mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \mapsto \mathcal{F}(\mathcal{L})$ fulfilling, for all $A, B \in \mathcal{F}(\mathcal{L})$, the following properties:

$\tilde{C}1)$ *fuzzy inclusion:* $A \subseteq \tilde{C}(A)$

$\tilde{C}2)$ *fuzzy monotony: if* $A \subseteq B$ *then* $\tilde{C}(A) \subseteq \tilde{C}(B)$

$\tilde{C}3)$ *fuzzy idempotence:* $\tilde{C}(\tilde{C}(A)) \subseteq \tilde{C}(A)$

where $(L, \wedge, \vee, 0, 1)$ is a complete lattice with minimum 0 and maximum 1 and $\mathcal{F}(\mathcal{L})$ denotes the the fuzzy power set (i.e. the set $L^{\mathcal{L}}$ of all fuzzy subsets) over a language $\mathcal{L}$.

On the other hand, Chakraborty extends in [4] the concept of consequence relation by defining Graded Consequence Relations as fuzzy relations between crisp sets of formulas and formulas.

Definition 2 (Graded Consequence Relation)
A fuzzy relation $g_c : \mathcal{P}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is called a Graded Consequence Relation if, for every $A, B \in \mathcal{P}(\mathcal{L})$ and $p, q \in \mathcal{L}$, g_c fulfills:

$g_c1)$ *fuzzy reflexivity:* $g_c(A, p) = 1$ *for all* $p \in A$

$g_c2)$ *fuzzy monotony: If* $B \subseteq A$ *then* $g_c(B, p) \leq g_c(A, p)$

$g_c3)$ *fuzzy cut:* $(\inf_{q \in B} g_c(A, q)) \otimes g_c(A \cup B, p) \leq g_c(A, p)$*, where* $\otimes$ *is a t-norm operation on* L

where $\mathcal{P}(\mathcal{L})$ denotes the classical power set over a language $\mathcal{L}$

In [8] Gerla examines the links between Fuzzy Closure Operators and Graded Consequence Relations. Castro et al. point out in [3] that several methods of approximate reasoning used in Artificial Intelligence are not covered by Graded Consequence Relations and they introduce a more general concept of Fuzzy Consequence Relation.

Definition 3 (Fuzzy Consequence Relation)
Any fuzzy relation $\tilde{g} : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is called a Fuzzy Consequence Relation if the following three properties hold for every $A, B \in \mathcal{F}(\mathcal{L})$ and $p, q \in \mathcal{L}$:

$\tilde{g}1)$ *fuzzy reflexivity:* $A(p) \leq \tilde{g}(A, p)$

$\tilde{g}2)$ *fuzzy monotony: If* $B \subseteq A$ *then* $\tilde{g}(B, p) \leq \tilde{g}(A, p)$

$\tilde{g}3)$ *fuzzy cut: If for all* p, $B(p) \leq \tilde{g}(A, p)$*, then for all* q, $\tilde{g}(A \cup B, q) \leq \tilde{g}(A, q)$

Notice that the Fuzzy Closure Operators and the Fuzzy Consequence Relations are related by the following properties:

- If $\tilde{C}$ is a Fuzzy Closure Operator then, $\tilde{g}(A,p) = \tilde{C}(A)(p)$ is a Fuzzy Consequence Relation.
- If $\tilde{g}$ is a Fuzzy Consequence Relation then, $\tilde{C}(A) = \tilde{g}(A,.)$ is a Fuzzy Closure Operator.

On the other hand a Fuzzy Closure Operator $\tilde{C}$ on $\mathcal{F}(\mathcal{L})$ induces a Closure System $\mathcal{C}$ on $\mathcal{F}(\mathcal{L})$ defined by

$$\mathcal{C} = \{T \in \mathcal{F}(\mathcal{L}) \mid \tilde{C}(T) = T\}.$$

And conversely, every closure system $\mathcal{C}$ in $\mathcal{F}(\mathcal{L})$ induces a fuzzy closure operator $\tilde{C}$ on $\mathcal{F}(\mathcal{L})$ defined by

$$\tilde{C}(A) = \bigwedge\{T \in \mathcal{C} \mid A \leq T\}.$$

In this paper we generalize, for fuzzy sets of formulas, Chakraborty's Graded Consequence Relations by defining what we call Implicative Consequence Relations, which generalize, at the same time, Castro et al.'s Fuzzy Consequence Relations. From these Implicative Consequence Relations we introduce and study their corresponding Implicative Closure Operators and Closure Systems. Finally we show that Implicative Consequence Relations also capture some well-known approximate entailments like the Approximation and Proximity entailments introduced by Dubois et al. in [5] and the Natural Inference Operator defined by Boixader and Jacas in [1].

2 Implicative Closure Operators

We are interested in a generalization of Graded Consequence Relations over fuzzy sets of formulas. Therefore, we introduce a new kind of fuzzy consequence relations that will be called *Implicative Consequence Relations*, since we will give the Fuzzy Cut property by means of an implication operation $\rightarrow$ on L. Thus we extend the lattice $(L, \wedge, \vee, 0, 1)$ to a complete BL-algebra[1], i.e. a structure $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a complete distributive lattice, $(L, \otimes, 1)$ is a commutative monoid, $(\otimes, \rightarrow)$ is a residuated pair, i.e. they verify

$$x \otimes y \leq z \text{ if and only if } x \leq y \rightarrow z,$$

and for all $x, y, z \in L$, it holds that $x \wedge y = x \otimes (x \rightarrow y)$ and $(x \rightarrow y) \vee (y \rightarrow x) = 1$. The operation $\rightarrow$ is usually called *residuated implication.*

[1] BL-algebras are introduced by Hájek in [9] as the algebraic counterpart of the so-called Basic Fuzzy Logic, which is the logic of continuous t-norms.

Definition 4 (Implicative Cons. Relation) *A fuzzy relation $\tilde{g}_c : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is called an Implicative Consequence Relation if, for every $A, B \in \mathcal{F}(\mathcal{L})$ and $p, q \in \mathcal{L}$, $\tilde{g}_c$ fulfills:*

$\tilde{g}_c$1) *fuzzy reflexivity:* $A(p) \leq \tilde{g}_c(A,p)$

$\tilde{g}_c$2) *fuzzy monotony: If* $B \subseteq A$ *then* $\tilde{g}_c(B,p) \leq \tilde{g}_c(A,p)$

$\tilde{g}_c$3) *fuzzy cut:* $[\inf_{q \in \mathcal{L}}(B(q) \rightarrow \tilde{g}_c(A,q))] \otimes \tilde{g}_c(A \cup B, p) \leq \tilde{g}_c(A,p)$ [2]

It is easy to prove that Graded Consequence Relations are a particular case of Implicative Consequence Relations. In fact, the main difference is to consider a graded fuzzy inclusion instead of an strict fuzzy inclusion in the cut property.

Implicative Consequence Relations also have a representation theorem which is a generalization of the one given in [4] for Graded Consequence Relations.

Theorem 1 *A fuzzy relation $\tilde{g}_c : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is an Implicative Consequence Relation if, and only if, there exists a family of fuzzy sets $\{T_i\}_{i \in I}$ satisfying $\tilde{g}_c(A,p) = \inf_i\{[\inf_{q \in \mathcal{L}} A(q) \rightarrow T_i(q)] \rightarrow T_i(p)\}$.*

The fuzzy sets $\{T_i\}_{i \in I}$ defining the Implicative Consequence Relation $\tilde{g}_c$, in the sense of the above theorem, will be called *generators* of $\tilde{g}_c$.

From the above theorem, it is easy to prove that any Implicative Consequence Relation satisfies the next additional property:

$\tilde{g}$4) $\tilde{g}_c(A \otimes \bar{k}, p) \geq \tilde{g}_c(A,p) \otimes k$,

where $\bar{k}$ is the constant fuzzy set such that all the elements of $\mathcal{L}$ belong to it with value k.

The Fuzzy Closure Operators corresponding to Implicative Consequence Relations, which we shall call them *Implicative Closure Operators*, are characterized by the properties: $\tilde{C}1, \tilde{C}2, \tilde{C}3$, and

$\tilde{C}$4) $\tilde{C}_c(A \otimes \bar{k})(p) \geq \tilde{C}_c(A)(p) \otimes \bar{k}$.

Property $\tilde{C}4$) is already mentioned by Gerla in [8] for the case when A is a crisp set and $\otimes = \min$. The following theorem characterizes *Implicative Closure System* corresponding to Implicative Closure Operators.

Theorem 2 *A closure system $\mathcal{C}_{\rightarrow}$ in $\mathcal{F}(\mathcal{L})$ is an Implicative Closure System if and only if, for any $E \in \mathcal{C}_{\rightarrow}$ and for any $k \in L$, $\bar{k} \rightarrow E \in \mathcal{C}_{\rightarrow}$.*

[2] This axiom could be also presented as $\tilde{g}_c(A \cup B, p) \leq (\inf_{q \in \mathcal{L}}(B(q) \rightarrow \tilde{g}_c(A,q))) \rightarrow \tilde{g}_c(A,p)$

We end this section by mentioning that Implicative Closure Operators satisfy a further property:

$\tilde{C}5)$ *Coherence*: $\tilde{C}_c(A)(q) \geq \tilde{C}_c(\{p\})(q) \otimes A(p)$,

where $\{p\}$ denotes a singleton, i.e. $\{p\}(p) = 1$ and $\{p\}(q) = 0$ otherwise. This property already appears in [2] where it is proved that if a closure operator satisfies $\tilde{C}5$ then $\tilde{C}_c(\{p\})(q)$ defines a fuzzy preorder in $\mathcal{L}$. Actually, it is easy to prove the following proposition.

Proposition 1 *If $\tilde{C}_c$ is an Implicative Closure Operator with generators $\{T_i\}_{i\in I}$, then it holds*

$$\tilde{C}_c(\{p\})(q) = \inf_{i\in I}\{T_i(p) \to T_i(q)\}.$$

Notice that the family $\{T_i\}_{i\in I}$ is a family of generators of the fuzzy preorder $R(p \mid q) = \tilde{C}_c(\{p\})(q)$.

3 Closure Operators defined by a fuzzy preorder

Different authors (see for instance [2]) have studied the so-called Fuzzy Closure Operators defined by a fuzzy preorder.

Definition 5 *Given a fuzzy preorder $R : \mathcal{L} \times \mathcal{L} \longmapsto L$ on the language $\mathcal{L}$, the associated Fuzzy Closure Operator $\tilde{C}_R$ is defined by:*

$$\tilde{C}_R(A)(q) = \bigvee_{p\in\mathcal{L}}\{R(q \mid p) \otimes A(p)\}.$$

The mapping $\tilde{C}_R$ provides, for every fuzzy set h , the least extensional fuzzy set (w.r.t. R) containing h, and it is proved in [10] that it satisfies the following properties:

- **C1**: $h \leq \tilde{C}_R(h)$
- **C2**: $\tilde{C}_R(\bigvee_{i\in I} h_i) = \bigvee_{i\in I} \tilde{C}_R(h_i)$
- **C3**: $\tilde{C}_R \circ \tilde{C}_R = \tilde{C}_R$
- **C4**: $\tilde{C}_R(\bar{k}) = \bar{k}$
- **C5**$^{\otimes}$: $\tilde{C}_R(\{q\} \otimes \bar{k}) = \tilde{C}_R\{q\} \otimes \bar{k}$
- **C6**$^{\otimes}$: $\tilde{C}(h \otimes \bar{k}) = \tilde{C}(h) \otimes \bar{k}$

where $\{q\}$ denotes a singleton and $\bar{k}$ the constant fuzzy set $\bar{k}(q) = k$, for all $q \in \mathcal{L}$.

The following remarks can be easily proved:

- **C2** implies fuzzy inclusion property
- in the presence of **C2**, **C5**$^{\otimes}$ is equivalent to **C6**$^{\otimes}$
- **C6**$^{\otimes}$ implies **C4**

In [7] Esteva et al. prove that a Fuzzy Closure Operator is generated by a fuzzy preorder (in the sense of above definition) if, and only if, it satisfies **C1-C5**$^{\otimes}$. On the other hand the closure system associated to these Fuzzy Closure Operators is the set of generators of the fuzzy pre-order R (see [10]). This closure system is characterized by the following properties:

1) it is closed under arbitrary unions, and

2) for any fuzzy set $E \in \mathcal{C}_{\to}$ and for any element $k \in L$, it holds that $\bar{k} \to E$, $\bar{k} \otimes E \in \mathcal{C}_{\to}$.

Taking into account this characterization and the above definition, the following theorem holds.

Theorem 3 *A Fuzzy Closure Operator defined by a fuzzy preorder R is the minimal Implicative Closure Operator $\tilde{C}_c$ such that $\tilde{C}_c(\{q\})(p) = R(q \mid p)$, for any $p, q \in \mathcal{L}$.*

4 Implicative Closure Operators and aproximate reasoning

In [6], given a similarity relation S on the set Ω of boolean interpretations of a propositional language $\mathcal{L}$, a fuzzy set p^* on Ω, called *approximately* p, is associated to each proposition $p \in \mathcal{L}$ in the following way:

$$p^*(w) = \sup_{w'\models p} S(w, w').$$

From this definition, Dubois et al. define in [5] two graded consequence relations on $\mathcal{L} \times \mathcal{L}$.

Definition 6 (Approximate Cons. Relation) *For each $p, q \in \mathcal{L}$, we define $p \models^{\alpha} q$ iff $I_S(q \mid p) = \inf_{w\models p} \sup_{w'\models q} S(w, w') \geq \alpha$.*

Definition 7 (Proximity Cons. Relation) *For each $p, q \in \mathcal{L}$ and each $K \subseteq \mathcal{L}$, we define $p \models^{\alpha}_{K} q$ iff $J_{S,K}(q \mid p) = \inf_{w\models K} p^*(w) \to q^*(w) \geq \alpha$.*

In [5] it is proved that $I_S(q \mid p) = J_{S,\top}(q \mid p)$, where $\top$ stands for a boolean tautology, i.e. any formula whose set of models is the whole set Ω. In our framework, the Approximate and Proximity Consequence Relations can be obtained as Implicative Closure Operators. Indeed, consider the family of fuzzy sets $F = \{\tilde{w}\}_{w\in\Omega}$, where for each $w \in \Omega$, the fuzzy set $\tilde{w} : \mathcal{L} \longmapsto [0, 1]$ is defined by $\tilde{w}(q) = q^*(w)$. Now define, for all $A \in [0, 1]^{\mathcal{L}}$,

$$\tilde{C}^*_c(A)(q) = \inf_{\tilde{w}\in F}(\inf_{p\in\mathcal{L}}(A(p) \to \tilde{w}(p))) \to \tilde{w}(q)).$$

An easy computation shows that $\tilde{C}_c^*(\{p\})(q) = J_{S,\top}(q \mid p) = I_S(q \mid p)$ and we obtain the Approximate Consequence Relation. Moreover, if we consider now the family of fuzzy sets to be $F = \{\tilde{w}\}_{w \models K}$, for a subset $K \subseteq \mathcal{L}$, then what we obtain is $\tilde{C}_c^*(\{p\})(q) = J_{S,K}(q \mid p)$, that is, the Proximity Consequence Relation.

In [1] Boixader and Jacas analyze "...approximate reasoning through extensionality with respect to the natural $\otimes$-indistinguishability operator, by considering the indistinguishability level between fuzzy sets as a formal measure of their degree of similarity,...". For this purpose, they introduce a family of operators $I : [0,1]^U \longmapsto [0,1]^V$, where U and V are universes of discourse. These operators are called *extensional inference operators* if they preserve the pointwise order, i.e. if $A_1 \leq_U A_2$ then $I(A_1) \leq_V I(A_2)$, and satisfy $\otimes$-extensionality, i.e. $\bar{E}_U^{\otimes}(A_1, A_2) \leq \bar{E}_V^{\otimes}(I(A_1), I(A_2))$, where $\bar{E}_U^{\otimes}$ is the similarity function defined by

$$\bar{E}_U^{\otimes}(A_1, A_2) = \inf_{x \in U} A_1(x) \vee A_2(x) \to A_1(x) \wedge A_2(x)$$

and analogously for $\bar{E}_V^{\otimes}$. Moreover, they prove that it is possible to associate to any fuzzy rule *"If A then B"* the so-called *Natural Inference Operator*, which is the optimal from the extensionality point of view.

Definition 8 (Natural Inference Operator) *Given the rule "If A then B" with $A \in [0,1]^U$ and $B \in [0,1]^V$, the Natural Inference Operator $\bar{I}_{AB} : [0,1]^U \longmapsto [0,1]^V$ associated to this rule is defined by*

$$\bar{I}_{AB}(A')(v) = [\inf_{u \in U}(A'(u) \to A(u))] \to B(v).$$

Theorem 4 *([1, Theorem 15]) The Natural Inference Operator associated to the rule "If A then B" is the least specific inference operator satisfying:*

- $\bar{I}_{AB}$ *is extensional,*
- $\bar{I}_{AB}(A) = B$,
- $\bar{I}_{AB}(A') \geq_V B$*, for any* $A' \in [0,1]^U$*. Moreover, if* $A' \leq_U A$*, then* $\bar{I}_{AB}(A') = B$

Now we are in position to show that Implicative Closure Operators $\tilde{C}_c$ are Natural Inference Operators. In fact, for any $\tilde{C}_c$ there exists a family $\{T_i\}_{i \in I}$ such that

$$\tilde{C}_c(A)(p) = \inf_{i \in I}\{[\inf_{q \in \mathcal{L}} A(q) \to T_i(q)] \to T_i(p)\}$$

If we take $U = V = \mathcal{L}$, it is obvious that, for each T_i, the operator I_i defined as

$$I_i(A)(p) = [\inf_{q \in \mathcal{L}} A(q) \to T_i(q)] \to T_i(p)$$

is the Natural Inference Operator on $\mathcal{L}$ associated to the rule *"if T_i then T_i"*, and thus the following theorem holds.

Theorem 5 *Implicative Closure Operators are Natural Inference Operators.*

Moreover, our representation theorem 1 should be seen as a particular case of the representation theorem in [1]. Thus, the operators $\tilde{C}_c$ are always extensional in the above sense. Finally, remark that closure operators coming from a $\otimes$-transitive fuzzy preorders are obviously $\otimes$-extensional. But, in general, we can only assure that closure operators are min-extensional.

Proposition 2 *Let $\tilde{C}$ be the closure operator defined by the family of fuzzy sets $\{T_i\}_{i \in I}$ by $\tilde{C}(A)(p) = \inf\{T_i(p) \mid A \subseteq T_i\}$. Then $\tilde{C}$ is* min*-extensional.*

However, this result is not true in general for t-norms different of min, indeed there are examples that are not Lukasiewicz-extensional.

References

[1] D. Boixader, J. Jacas (1998) Extensionality based approximate reasoning. International Journal of Approximate Reasoning",19, pag. 221-230.

[2] J.L. Castro, E. Trillas (1999). Tarki's fuzzy consequence. Submitted to IJAR

[3] J.L. Castro , E. Trillas , S. Cubillo (1994). On consequence in approximate reasoning. Journal of Applied Non-Classical Logics, vol. 4 nro. 1, pp. 91-103.

[4] M.K. Chakraborty (1988). Use of fuzzy set theory in introducing graded consequence in multiple valued logic. Fuzzy Logic in Knowledge Systems, Decision and Control, M.M. Gupta and T.Yamakawa (eds). North Holland, Amsterdam, pp. 247-257.

[5] D. Dubois, F. Esteva, P. Garcia, L. Godo, H. Prade (1997). A Logical approach to interpolation based on similarity relations. Int. J. of Approximate Reasoning, 17 (1), 1-36.

[6] F. Esteva, P. Garcia, L. Godo (1994). Relating and extending semantical approaches to possibilistic reasoning. Int. J. of Approximate Reasoning, 11, 312-344.

[7] F. Esteva, P. Garcia, L. Godo, R.O. Rodríguez (1998). Fuzzy Approximation Relations, Modal Structures and Possibilistic Logic. Mathware & Softcomputing, 5, n. 2-3, 151-166.

[8] G. Gerla (1996). Graded Consequence Relations and Fuzzy Closure Operators. Journal of Applied Non-Clasical Logics, vol. 6 nro. 4, 369-379.

[9] P. Hájek, Methamatematics of Fuzzy Logic, Kluwer, 1998.

[10] J. Jacas, J. Recasens (1995). Fuzzy T-transitive relations: eigenvectors and generators. Fuzzy Sets & Systems 72, 147-154.

[11] J. Pavelka (1979) On the fuzzy logic I: Many valued rules and inference. Math. Logik Grundlag. Math. 25, 45-52.