ON (S,N)-IMPLICATIONS IN FUZZY LOGIC CONSISTENT WITH T-CONJUNCTIONS

C. Alsina
Secció de Matemàtiques i Informàtica,
Univ. Politècnica de Catalunya
Avda. Diagonal, 649. 08028 Barcelona
alsina@ea.upc.es

E. Trillas
Depto. Inteligencia Artificial,
Universidad Politécnica de Madrid
28660 Boadilla del Monte (Madrid)
etrillas@fi.upm.es

Summary

Motivated by some fundamental relations in Fuzzy Logic we study some basic cases of the functional equation $S_1(x,y) = S_2(x, T(N(x),y))$. In doing this some interesting techniques for solving functional equations are faced.

Key words: (S,N)-implications, t-norms, fuzzy connectives.

1 INTRODUCTION

In the context of Fuzzy Logic one considers frequently a triple of logical connectives *and, or* and *not* by introducing the functions (T, S, n) where T is a continuous t-norm, S is a continuous t-conorm and n is a strong negation. In this framework the modeling of fuzzy implications sometimes is made by means of the so-called S-implications, namely the functions

$$I(y|x) = S\left(n(x), y\right),$$

generalizing the classical identification of a conditional statement 'If p, then q' (represented by $p \to q$) with the statement 'Not p or q' represented by $\neg p \vee q$). When the classical setting is that of a Boolean Algebra B, the identity

$$p \to q = \neg p \vee q - \neg p \vee (p \wedge q) = p \to p \wedge q$$

enjoys the property $\neg p \wedge (p \wedge q) = 0$ that allows to some important laws of any probability P on B. For example, the proof that P is a non-decreasing function fro B to [0,1] **should** be obtained as follows: provided that $p \geq q$ (in the partial order of B) it is $q = p \vee q = p \vee (\neg p \wedge q)$ hence, in the linear order of $[0,1]$, it is $P(q) = P(p) + P(\neg p \wedge q) \geq P(p)$, because of the additivity of P. Consequently, it seems interesting to know when the identity $p \to q = p \to p \wedge q$ holds in a Fuzzy Logic on which the implication is modelled by an S-implication. In this context let's consider the following

Definition 1.1. Let $I_1(|)$ be a (S_1, n_1)-implication and let $I_2(|)$ be a (S_2, n_2)-implication. Fixed a conjunction T defined by a t-norm we will say that I_2 is T-**consistent** with I_1 whenever $I_2(T(x,y)|x) = I_1(y|x)$, i.e.,

$$S_2\left(n_2(x), T(x,y)\right) = S_1\left(n_1(x), y\right).$$

This equation corresponds to the fact that "$p \Rightarrow q$" and "$p \Rightarrow p \wedge q$" are conveniently identified.

Note that setting $y = 0$ yields $n_1 = n_2$. So we will consider only one negation $N = n_1 = n_2$ and we will take care of the condition

$$S_2\left(N(x), T(x,y)\right) = S_1\left(N(x), y\right). \qquad (1)$$

Next note that from (1) with $y = 1$ it follows that necessarily $S_2(N(x), x) = 1$, i.e., $I_2(x|x) = 1$ but (1) says nothing about the possible values of $I_1(x|x) = S_1\left(N(x), x\right)$.

Our aim in this paper is to solve completely (1) when $S_1(x,y) < 1$ whenever $x, y \neq 1$ and to study (1) also in some other related cases.

In doing this study we extend previous results found by us in [4] where equation (1) was considered in the case where N was precisely the strong negation associated to S_2. Here we solve (1) just with the restriction $S_1(x,y) < 1$ whenever $x, y \neq 1$. The other cases for S_1 will be treated elsewhere. In doing this we have developed some functional equations techniques that have their own mathematical interest. We assume notations and concepts as given in [3,9].

2 STUDY OF EQUATION (1)

Our aim here is to solve (1) when S_1 is a continuous t-conorm such that $S_1(x,y) < 1$ whenever, $x, y \neq 1$,

S_2 is a continuous t-conorm, T is a continuous t-norm and N is a strong negation. As we have seen before it follows from (1) that $S_2(N(x), x) = 1$ so S_2 will be a non-strict Archimedean t-conorm representable in the form

$$S_2(x, y) = s_2^{(-1)}\left(s_2(x) + s_2(y)\right), \quad (2)$$

where $s_2 : [0, 1] \to [0, 1]$ is continuous, strictly increasing, $s_2(0) = 0$, $s_2(1) = 1$ and $s_2^{(-1)}(x) = s_2^{-1}(x)$ for x in [0,1], $s_2^{(-1)}(x) = 1$ for $x \geq 1$. Thus S_2 has its associated strong negation

$$N_2(x) = s_2^{-1}(1 - s_2(x)), \quad (3)$$

and $S_2(x, y) = 1$ if and only if $y \geq N_2(x)$. In particular since $S_2(x, N(x)) = 1$ we need to have $N \geq N_2$. Note that (1) is a functional equation of Pexider type with two variables x, y and four unknown functions S_2, S_1, T, N.

Lemma 2.1. Let S_2 and N_2 be given by (2) and (3), respectively. Let N be a strong negation and let T be a continuous t-norm satisfying

$$S_2\left(x, T(N(x), y)\right) = \mathrm{Max}(x, y), \quad (4)$$

for all x, y in [0,1]. This is possible if and only if T is a non-strict Archimedean t-norm with additive generator $t(x) = 1 - s_2(x)$ (i.e., T is the N_2-dual of $S_2 : T = N_2 \circ S_2 \circ N_2 \times N_2$) and $N = N_2 = N_T$.

Lemma 2.2. If S_2 and N_2 are given by (2) and (3), respectively, N is a strong negation, T is a continuous t-norm, S_1 is a continuous t-conorm with $S_1(x, y) < 1$ whenever $x, y \neq 1$ and (1) holds, then necessarily $N = N_2$ and T must have one of the following forms:

(i) $T = s_2^{-1} \circ \mathrm{Prod} \circ s_2 \times s_2$

(ii) $T = s_2^{-1} \circ T_\lambda \circ s_2 \times s_2$, where T_λ belongs to Frank's family.

(iii) $T = s_2^{(-1)} \circ W \circ s_2 \times s_2$;

(iv) T =Min or T is an ordinal sum of Archimedean t-norms of the above type.

Proof. The commutativity of S_1 and (1) yield at once

$$\begin{aligned} S_1(x, y) &= S_2(x, T(N(x), y)) = S_2(y, T(N(y), x)) \\ &= S_1(y, x), \end{aligned}$$

and since we are assuming $S_1(x, y) < 1$ for $x, y \neq 1$ we deduce, using (2), that

$$s_2(x) + s_2(T(N(x), y)) = s_2(y) + s_2(T(N(y), x)). \quad (5)$$

Introduce the new variables $u = s_2(y)$ and $v = s_2(N(x))$ into (6). Then for u, v in [0,1] arbitrary we obtain

$$\begin{aligned} &\left(s_2 \circ N \circ s_2^{-1}\right)(v) + s_2\left(T(s_2^{-1}(v), s_2^{-1}(u))\right) = \\ &= u + \left(s_2 \circ N \circ s_2^{-1} \circ s_2 \circ N\right) \\ &\left(T(N(s_2^{-1}(u)), N(s_2^{-1}(v))\right) \end{aligned} \quad (6)$$

Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(t) = (s_2 \circ N \circ s_2^{-1})(t). \quad (7)$$

Then f is strictly decreasing continuous and $f(0) = 1$, $f(1) = 0$. Consider now the t-norm:

$$G(u, v) = s_2\left(T(s_2^{-1}(u), s_2^{-1}(v))\right) \quad (8)$$

and the t-conorm

$$H(u, v) = (s_2 \circ N)\left[T\left((N \circ s_2^{-1}(u), (N \circ s_2^{-1})(v)\right)\right]. \quad (9)$$

By virtue of the equation (7)

$$f(v) + G(u, v) = u + H(u, v),$$

therefore changing the roles of u, v we must have also

$$f(u) + G(u, v) = v + H(u, v),$$

whence substraction of the last equalities yield

$$f(v) - f(u) = u - v,$$

i.e.,

$$f(v) + v = f(u) + u,$$

and $f(x) + x$ is a constant function: $f(x) + x = k$. The condition $f(0) = 1$ yields $k = 1$, i.e., $f(x) = 1 - x$, whence by (8), $N(x) = s_2^{-1}(1 - s_2(x)) = N_2(x)$.

Now with $N = N_2$ we go back to (6) to get

$$\begin{aligned} &s_2(x) + s_2\left(T\left(s_2^{-1}(-s_2(x)), y\right)\right) = \\ &s_2(y) + s_2\left(T(s_2^{-1}(1 - s_2(y), x)\right). \end{aligned} \quad (10)$$

The substitutions $a = s_2(x)$, $b = 1 - s_2(y)$ yield

$$\begin{aligned} &u + s_2\left[T(s_2^{-1}(1 - u), s_2^{-1}(1 - v))\right] = \\ &1 - v + s_2\left[T(s_2^{-1}(v), s_2^{-1}(u))\right]. \end{aligned} \quad (11)$$

Thus the t-norm $T_2(u, v) = s_2\left[T(s_2^{-1}(u), s_2^{-1}(v)))\right]$ and its associated t-conorm

$$T_2^*(u, v) = 1 - T_2(1 - u, 1 - v)$$

would satisfy by virtue of (12) Frank's equation

$$T_2(u, v) + T_2^*(u, v) = u + v,$$

i.e., T_2 is completely determined by Frank's theorem and so is T. The lemma follows.

Lemma 2.3. If S_2 and N_2 are given by (2) and (3), respectively, N is a strong negation, T is a continuous t-norm, S_1 is a strict t-conorm and (1) holds, then $N = N_2$ and there are two types or solutions either

(a) $T = s_2^{-1} \circ \text{Prod} \circ s_2 \times s_2$; $S_1 = s_2^{-1} \circ \text{Prod}^* \circ s_2 \times s_2$,

or

(b) $T = s_2^{-1} \circ T_\lambda \circ s_2 \times s_2$; $S_1 = s_2^{-1} \circ T^*_{1/\lambda} \circ s_2 \times s_2$;

where T_λ belongs to Frank's family (see Theorem 2.2.).

Proof. If S_1 is a strict t-conorm, $S_1(x,y) < 1$ whenever $x \cdot y \neq 1$ and we can apply Lemma 2.2, so $N = N_2$ and the possibles forms of the t-norm T are determined. Then checking which operations one can be consistent with the fact that (1) holds for a strict t-conorm S_1 we obtain the lemma.

Lemma 2.4. If S_2 and N_2 are given by (2) and (3), respectively, T is a continuous t-norm, N is a strong negation, S_1 is an ordinal sum of Archimedean t-conorms and (1) holds then $S_2(x,y) < 1$ for all $x, y \neq 1$.

Proof. Under the above conditions consider the set $O = \{(x,y) \in [0,1]^2 \mid S_1(x,y) = 1\}$. Since S_1 is an ordinal sum, if there would exist a point $(x_0, y_0) \in O$ with $x_0, y_0 \neq 1$ then S, would take the value 1 in a region of the form

$$P = \{(x,y) \mid x \in [b,1],\ y \geq g(x)\} \subset O,$$

for some b in (0,1) idempotent element of S_1 and for some function $g : [b,1] \to [b,1]$ continuous, strictly decreasing, $g(b) = 1$, $g(1) = b$ and $g = g^{-1}$.

When (x,y) moves in P the corresponding point $(N(x), y)$ moves in

$$Q = \{(a,b) \mid 0 \leq a \leq N(b),\ b \geq g(N(a))\}$$

and in this case $(x,y) \in P$ and (1) imply $1 = S_1(x,y) = S_2(x, T(N(x),y))$, i.e., $T(N(x),y) \geq N_2(x)$, inquality which implies in particular that T can not vanish in Q.

Next note that we must have S_1 =Max on $[b,1] \times [0,b] \cup [0,b] \times [b,1]$ whence for $0 \leq y \leq b \leq x \leq 1$

$$x = \text{Max}(x,y) = S_1(x,y) = S_2(x, T(N(x),y)),$$

and necessarily $T(N(x),y) = 0$, but when (x,y) moves in $[b,1] \times [0,b]$ the points $(N(x),y))$ move in $K = [0, N(b)] \times [0,b]$, i.e., $T \equiv 0$ on K. Recalling the representation theorem, T can not be Min or a strict t-norm because T vanishes on K, T can not be a non-strict Archimedean t-norm because T does not vanish in Q,... and T can not be an ordinal sum because its vanishing Archimedean component would need to include Q in its zero set and therefore its zero set would need to cut Q efectively, which is not possible. We get a contradiction and the lemma is proved. □

Lemma 2.5. If S_2 and N_2 are given by (2) and (3), respectively, N is a strong negation and S_1 is an ordinal sum of Archimedean t-norms then there exists no continuous t-norm satisfying (1).

From all the above lemmata it follows at once our main result

Theorem 2.1. Let S_2 be a non-strict Archimedean t-conorm with additive generator s_2 and let N_2 be its associated strong negation (3). Let N be a strong negation. Then a continuous t-norm T and a continuous t-conorm S_1 such that $S_1(x,y) < 1$ for $x, y \neq 1$, satisfy (1) if and only if one of the following conditions hold:

(i) S_1 =Max, $T = N_2 \circ S_2 \circ N_2 \times N_2$, $N = N_2 = N_T$;

(ii) $S_1 = s_2^{-1} \circ \text{Prod}^* \circ s_2 \times s_2$, $T = s_2^{-1} \circ \text{Prod} \circ s_2 \times s_2$, $N = N_2$;

(iii) $S_1 = s_2^{-1} \circ T^*_{1/\lambda} \circ s_2 \times s_2$, $T = s_2^{-1} \circ T_\lambda \circ s_2 \times s_2$; $N = N_2$;

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