Fuzzy projective geometries

Leen Kuijken*

Department of Applied Mathematics and Computer Science
University of Ghent, Krijgslaan 281 (S9),
B-9000 Gent, BELGIUM
lk@cage.rug.ac.be

Summary

In this paper we introduce a new model of fuzzy projective geometries. This model is richer than the one introduced in [4] and [5], since points and lines play a similar role, like they do in classical projective geometry. Furthermore, we will show that this new fuzzy projective geometry is closely related to the fibred projective geometries introduced in [6].
Keywords: Fuzzy geometry, Fibred geometry.

1 Introduction

In [4] and [5] we introduced a first model of fuzzy projective geometries, deduced from respectively fuzzy vector spaces and fuzzy groups. This provided a link between the fuzzy versions of classical theories that are very closely related. However, the geometric structure involved in this model is rather weak: a fuzzy projective space in this sense is equivalent with a given sequence of subspaces in the base projective space.
In [6] another fuzzy model of projective geometries was constructed: fibred projective planes. In this model the role of points and lines is equivalent (this is not the case in the first model), as in the classical case. Points and lines of the base geometry mostly have multiple degrees of membership.
This paper introduces a third model. We first define a fuzzy projective plane in which points and lines play the same role, and such that every point and every line in the base plane possess only one degree of membership. Afterwards we give a definition for an n-dimensional fuzzy projective space. We also investigate the link between fibred and fuzzy projective geometries.

*The author is a Research Fellow supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), grant No. 961336

2 Preliminaries

Definition 2.1 ([2])
An **(axiomatic) projective plane** $\mathcal{P}$ is an incidence structure (P, B, I) with P a set of points, B a set of lines and I an incidence relation, such that the following axioms are satisfied:
(A1) every pair of distinct points are incident with a unique common line;
(A2) every pair of distinct lines are incident with a unique common point;
(A3) $\mathcal{P}$ contains a set of four points with the property that no three of them are incident with a common line.
A **closed configuration** S of $\mathcal{P}$ is a subset of $P \cup B$ that is closed under taking intersection points of any pair of lines in S and lines spanned by any pair of distinct points of S. We denote the line in $\mathcal{P}$ spanned by the points a and b by $\langle a, b\rangle$.

Definition 2.2 ([1], p. 10)
An **(axiomatic) projective space** $\mathcal{S}$ is an incidence structure (P, B, I) with P a set of points, B a set of lines and I an incidence relation, such that the following axioms are satisfied:
(A1) every line is incident with at least two points;
(A2) every pair of distinct points are incident with a unique common line;
(A3) given distinct points a, b, c, d, e such that $\langle a, b\rangle = \langle a, c\rangle \neq \langle a, d\rangle = \langle a, e\rangle$, there is a point $xI\langle b, d\rangle \cap \langle c, e\rangle$ (Pasch's axiom).

Definition 2.3 ([7])
A **fuzzy set** λ on a set X is a mapping $\lambda : X \rightarrow [0, 1]$:

$x \mapsto \lambda(x)$. The number $\lambda(x)$ is called the **degree of membership** of the point x in λ.

We denote the minimum operator by $\wedge$.

Definition 2.4 ([4], [5])
A fuzzy set λ on the n-dimensional projective space $\mathcal{S}$ is a **fuzzy n- dimensional projective space** on $\mathcal{S}$ if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$, for any three collinear points p, q, r of $\mathcal{S}$. We denote $[\lambda, \mathcal{S}]$.
The projective space $\mathcal{S}$ is called the **base projective space** of $[\lambda, \mathcal{S}]$. If $\mathcal{S}$ is a fuzzy point, line, plane, ..., we use base point, base line, base plane, ..., respectively.

Definition 2.5 ([6])
Consider the projective plane $\mathcal{P} = (P, B, I)$. Suppose $a \in P$ and $\alpha \in]0,1]$. The **f-point** (a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:

$$\begin{array}{rccl} (a,\alpha): & P & \rightarrow & [0,1] \\ & a & \mapsto & \alpha \\ & x & \mapsto & 0 \qquad \text{if } x \in P \backslash \{a\}. \end{array}$$

The point a is called the **base point** of the f-point (a, α). An **f-line** (L, α) with base line L is defined in a similar way.

With the definition of the intersection of fuzzy sets (see [7]) and the extension principle (see [3]), we see that:

Definition 2.6 ([6])
The f-lines (L, α) and (M, β) intersect in the unique f-point $(L \cap M, \alpha \wedge \beta)$.
The f-points (a, λ) and (b, β) span the unique f-line $(\langle a, b \rangle, \lambda \wedge \beta)$.

Definition 2.7 ([6])
A **fibred projective plane** $\mathcal{FP}$ on the projective plane $\mathcal{P}$ consist of a set $\mathcal{F}P$ of f-points and a set $\mathcal{F}B$ of f-lines, such that every point and line of $\mathcal{P}$ is base point and base line of at least one f-point and f-line respectively, and such that $(\mathcal{F}P, \mathcal{F}B)$ satisfies the following fuzzified axioms of a projective plane (cf. definition 2.1):
(F1) every pair of f-points with distinct base points span a unique f-line;
(F2) every pair of f-lines with distinct base lines intersect in a unique f-point;
The projective plane $\mathcal{P}$ is called the **base geometry** of $\mathcal{FP}$.

We can construct a fibred projective plane in the following way (see [6]). Let $P' \subseteq P$ and $B' \subseteq B$ be such that the unique closed configuration containing $P' \cup B'$ is $P \cup B$. For each element x of $P' \cup B'$, we choose arbitrarily a nonempty subset Σ_x of $]0,1]$ of which the elements are called the **initial values of** x, and we define a fibred projective plane $\mathcal{FP}$ as follows. For each $x \in P' \cup B'$ and for each $\alpha \in \Sigma_x$, the element (x, α) belongs to $\mathcal{FP}$. This is step 1 of the construction. We now describe step i, $i > 1$.

For any pair of f-points that we already obtained, the f-line spanned by it also belongs to $\mathcal{FP}$ by definition. Dually, for any pair of f-lines, the intersection f-point belongs to $\mathcal{FP}$. The set of all f-points and of all f-lines constructed this way in a finite number of steps is readily verified to constitute a fibred projective plane.

It is clear that every fibred projective plane can be constructed as above. Indeed, one can always take for each element all its corresponding values as initial values.

Now suppose Σ_x is a singleton for every $x \in P \cup B$. If $P' = P$ and $B' = \emptyset$, then we call the fibred projective plane **mono-point-generated**. If $P' = P$ and $B' = B$, then the fibred projective plane is called **mono-generated**. We will restrict ourselves to these two kinds of fibred projective planes.

We see that $\mathcal{FP}$ can be considered as an ordinary projective plane (its base plane $\mathcal{P}$) where to every point and line, a set of values from $]0,1]$ are assigned. Also the fuzzy projective plane in definition 2.4 can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degree of membership is assigned.

Example: (see [6])
Consider the classical projective plane $\mathcal{F} = GF(2,2)$, the Fano plane. We will construct a mono-point-generated fibred projective plane with base plane $\mathcal{F}$. We label the 7 points of $\mathcal{F}$ as $\{a, b, c, d, e, f, g\}$ and the lines as $\{A, B, C, D, E, F, G\}$, such that:
$A = \{a, b, c\}$,
$B = \{c, d, e\}$,
$C = \{e, f, a\}$,
$D = \{a, g, d\}$,
$E = \{b, g, e\}$,
$F = \{c, g, f\}$,
$G = \{b, d, f\}$.

In step 1, we construct the f-points $(a, 0.9), (b, 0.8), (c, 0.7), (d, 0.6), (e, 0.3), (f, 0.4)$ and $(g, 0.5)$ on the points of P, thus 0.9, 0.8, 0.7, 0.6, 0.3, 0.4, and 0.5 are the initial values of the resp. base points a, b, c, d, e, f and g. Following the foregoing construction, these initial values yield the following fibred projective plane:

$\Sigma_a = \{0.3, 0.4, 0.5, 0.6, 0.9\}$,
$\Sigma_b = \{0.3, 0.4, 0.5, 0.6, 0.8\}$,
$\Sigma_c = \{0.3, 0.4, 0.5, 0.6, 0.7\}$,
$\Sigma_d = \{0.3, 0.4, 0.5, 0.6\}$,
$\Sigma_e = \{0.3, 0.4, 0.5\}$,
$\Sigma_f = \{0.3, 0.4, 0.5\}$,
$\Sigma_g = \{0.3, 0.4, 0.5\}$, and

$\Sigma_A = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$,
$\Sigma_B = \{0.3, 0.4, 0.5, 0.6\}$,
$\Sigma_C = \{0.3, 0.4, 0.5\}$,
$\Sigma_D = \{0.3, 0.4, 0.5, 0.6\}$,
$\Sigma_E = \{0.3, 0.4, 0.5\}$,
$\Sigma_F = \{0.3, 0.4, 0.5\}$,
$\Sigma_G = \{0.3, 0.4, 0.5, 0.6\}$.

3 Fuzzy projective planes

In this section we introduce a third model of a fuzzy projective geometries. Like in the fibred model, it also assigns values to the lines of the base geometry. Like the model in definition 2.4 it assigns only *one* value to every point (and line) of the base geometry, as custom in most definitions in the area of fuzzy algebra. In the sequel, a fuzzy projective plane will refer to the following definition, no longer to definition 2.4, so there will be no confusion.

Definition 3.1 Suppose $\mathcal{P}$ is a projective plane (P, B, I). The fuzzy set μ on $P \cup B$ is a **fuzzy projective plane** on $\mathcal{P}$ if
(1) $\mu(L) \geq \mu(p) \wedge \mu(q)$, $\forall p, q : \langle p, q\rangle = L$ and
(2) $\mu(p) \geq \mu(L) \wedge \mu(M)$, $\forall L, M : L \cap M = p$.

Definition 3.2 Consider the fibred projective plane $\mathcal{FP}$ with base plane the projective plane $\mathcal{P} = (P, B, I)$. As in section 2, let Σ_p (resp. Σ_L) be the set of all different degrees of membership of a point p (resp. line L), for all $p \in P$ and $L \in B$. **Skimming** $\mathcal{FP}$ means that for every element x of $\mathcal{P}$, we only keep the highest degree of membership, i.e. $\sup \Sigma_x$. This results in a fuzzy set χ on the base projective plane $\mathcal{P}$, called the **cream** of the fibred projective plane $\mathcal{FP}$. Thus:

$$\begin{array}{llll} \chi: & P \cup B & \to & [0,1] \\ & x & \mapsto & \sup \Sigma_x \end{array}$$

Theorem 3.1 *The cream of a fibred projective plane is a fuzzy projective plane, and every fuzzy projective plane can be considered as the cream of a fibred projective plane.*

This theorem makes sure the new definition makes sense: since fibred projective planes exist, fuzzy projective planes will also exist.

Example:
By the previous theorem, we know that the example in section 2 gives rise to the following fuzzy projective plane μ on the Fano plane $\mathcal{F}$:
$\mu(a) = 0.9$, $\mu(b) = 0.8$, $\mu(c) = 0.7$, $\mu(d) = 0.6$, $\mu(e) = \mu(f) = \mu(g) = 0.5$ and
$\mu(A) = 0.8$, $\mu(B) = 0.6$, $\mu(C) = 0.5$, $\mu(D) = 0.6$, $\mu(E) = \mu(F) = 0.5$, $\mu(G) = 0.6$.

4 Fuzzy projective spaces

So far we have only considered 2-dimensional fibred and fuzzy projective geometries, i.e. the plane case. We can also define n-dimensional fibred and fuzzy projective geometries, with n an arbitrary finite integer, such that the previous theorem holds in the general case. Consider the n-dimensional projective space $\mathcal{S}$. Call U_i the set of all i-dimensional subspaces of $\mathcal{S}$, for all i: $0 \leq i \leq n-1$.

Definition 4.1 Suppose $\mathcal{S}$ is an n-dimensional projective space, and $i < n$. An **f-subspace** (V_i, α) **of dimension** i is the following fuzzy set on the set U_i of $\mathcal{S}$:

$$\begin{array}{lllll} (V_i, \alpha): & U_i & \to & [0,1] & \\ & V_i & \mapsto & \alpha & \\ & x & \mapsto & 0 & \text{if } x \in U_i \backslash \{V_i\}. \end{array}$$

The subspace V_i is the **base subspace** of (V_i, α).

Definition 4.2 An n-dimensional **fibred projective space** $\mathcal{FS}$ on the n-dimensional projective space $\mathcal{S}$ consist of n sets of f-objects: f-subspaces of dimension i, for $0 \leq i \leq n-1$. Every subspace (of dimension i) of $\mathcal{S}$ is base subspace of at least one f-subspace (of dimension i). Moreover the following axioms have to be fullfilled:

(F1) the intersection of two f-subspaces (with distinct base subspaces that are not disjoint) is again an f-subspace;

(F2) every two f-subspaces (with distinct base subspaces that do not span $\mathcal{S}$ itself) span an f-subspace.

For $i = 0, 1, 2, n-1$, the f-subspaces of dimension i will be called f-points, f-lines, f-planes and f-hyperplanes.

The **cream** of an n-dimensional fibred projective space is defined in the same way as for a fibred projective plane (see definition 3.2)

Definition 4.3 Suppose $\mathcal{S}$ is an n-dimensional projective space as defined above. The fuzzy set μ on $\bigcup_{i=0}^{n-1} U_i$ is a **fuzzy projective space** of dimension n on $\mathcal{S}$ if for all subspaces V_i, V_j, V_k, $0 \leq i,j,k \leq n-1$ we have:
(1) $\mu(V_i) \geq \mu(V_j) \wedge \mu(V_k)$, $\forall\, V_j$ and V_k, $V_j \neq V_k$ such that $V_j \cap V_k = V_i$ if $V_i \neq \emptyset$ and
(2) $\mu(V_i) \geq \mu(V_j) \wedge \mu(V_k)$, $\forall\, V_j$ and V_k, $V_j \neq V_k$ such that $\langle V_j, V_k \rangle = V_i$ if $V_i \neq \mathcal{S}$.

Theorem 4.1 *The cream of an n-dimensional fibred projective space is an n-dimensional fuzzy projective space, and every n-dimensional fuzzy projective space can be considered as the cream of an n-dimensional fibred projective space.*

Acknowledgment
This results are part of the research done for my PhD thesis, supervised by Prof. H. Van Maldeghem and Prof. E. Kerre.

References

[1] F. Buekenhout, *Handbook of Incidence Geometry, Buildings and Foundations*, North – Holland, Amsterdam, 1995.

[2] D. R. Hughes and F. C. Piper, *Projective Planes*, Springer – Verlag, New York Heidelberg Berlin, 1973.

[3] E. E. Kerre (Ed.), *Introduction to the basic principles of fuzzy set theory and some of its applications*, Communication & Cognition, Gent/Belgium, 1993.

[4] L. Kuijken, H. Van Maldeghem and E. Kerre, *Fuzzy Projective Geometries from Fuzzy Vector Spaces*, Proceedings of IPMU '98, 1331 – 1338.

[5] L. Kuijken, H. Van Maldeghem and E. Kerre, *Fuzzy Projective Geometries from Fuzzy Groups*, Tatra Mountains Mathematical Publications, to appear.

[6] L. Kuijken, H. Van Maldeghem, *Fibred geometries*, Discrete Mathematics, submitted for publication.

[7] L. Zadeh, Fuzzy Sets, *Information and control* **8** (1965), 338–358.