PROPERTIES OF THE FUZZY COMPOSITIONS BASED ON AGGREGATION OPERATORS AND ITS PSEUDOCOMPLEMENT

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Abstract

The main aim of this paper is to present the properties of the $sup - F$ composition based on the aggregation operator F defined by Mayor and Torrens and analyze some properties about the pseudocomplement φ and the $inf - \varphi$ composition, to show that these compositions have analogous properties to the based on t-norms.

Keywords: Aggregation operators, composition of fuzzy relations, t-norms, t-relative pseudocomplement.

1 INTRODUCTION

The problem of finding appropriate connectives for the logical combination of fuzzy sets and fuzzy relations has turned to be an important issue from several points of view. In this context, triangular norms and conorms have shown to be useful tools in multiple fields, and more recently the aggregation operators play also an important role in fuzzy sets theory and in applied areas as social choice, multicriteria decision making, synthesis of implication functions, etc. But apart from the applications of these aggregation operators in the fields above mentioned, there is an increasing interest in their theoric study, which no doubt is going to lead to new possibilities of applications.

The aggregation has been defined by many authors as fuzzy operators generalizing "*and*" and "*or*" fuzzy connectives ([8], [11], [12], [14], [16], [22]) and has been applied as a consensus of fuzzy opinions in several contexts ([5], [6], [9], [10], [12], [13], [17], [19], [23]). Moreover nearly each author has proposed his own definition. Anyway, we can state that the axioms for the definition of these aggregation operators F, defined in the unit interval, whether binary or n-ary with $n \geq 3$, usually requires that F must be commutative, non-decreasing in each place and verify the boundary conditions $F(0, ..., 0) = 0$ and $F(1, ..., 1) = 1$. In this respect, the binary aggregations of Cholewa [6], Dombi [8], Fung & Fu [11], Mizumoto [16] and Zimmermann & Zysno [23], and the n-ary aggregations of Dubois & Koning [9], Fodor & Roubens [10], Klir & Folger [12], Montero [17], Ramakrishnan & Rao [19] and Yager [22] verify these conditions.

One of the aggregation operators which verify these properties has been introduced by Mayor & Torrens [14] as a generalization of t-norms and t-conorms, and it is given in the next definition.

Definition 1. *An aggregation operator is a commutative binary operation F in $[0,1]$, non-decreasing in each place, such that*

$$F(x, 0) = kx \qquad F(x, 1) = (1-k)x + k$$

for every $x \in [0,1]$, where $k = F(0,1)$.

This operator includes both t-norms and t-conorms and even any convex linear combination of a t-norm and a t-conorm. In this respect, Alsina, Mayor, Tomas & Torrens have proved in [1] that F is a t-norm if and only if it is associative and $k = 0$, and also that F is a t-conorm if and only if it is associative and $k = 1$. Moreover, we have that F is associative if and only if it is a t-norm (and then $k = 0$) or a t-conorm (and then $k = 1$).

In this context, we study in the next section the $sup - F$ composition based in the aggregation operator, as a generalization of $sup - t$ composition based on t-norms t. Moreover, the main aim of the sections 3 and 4 is to define the pseudocomplement φ associated with the aggregation operator and the $inf - \varphi$ composition and analyze their properties, as an extension of the t-relative pseudocomplement analized by Weber [21], Miyakoshi and Shimbo [15] and Klir and Yuan [12].

2 SUP-F COMPOSITION OF FUZZY RELATIONS

It is well known that the $sup - t$ composition of binary fuzzy relations, where t refers to a t-norm, generalize the standard $max - min$ composition and the $max - *$ composition (analyzed by Bezdek & Harris [2] and Rosenfeld [20]). In the same way, from the above binary aggregation F, we can define the $sup - F$ composition of two fuzzy binary relations as we defined in [4] and [3]. This $sup - F$ composition generalize also the standard $max - min$ composition, the $max - *$ composition and even the $sup - t$ composition.

Let X, Y be two nonempty sets and $[0,1]$ be the lattice with the usual "$\vee$" and "$\wedge$" operators. If we denote by $[0,1]^{X \times Y}$ the set of all the fuzzy binary relations, i.e., the fuzzy sets in the product space $X \times Y$, then we can define the $sup - F$ composition as follows:

Definition 2. *The $sup - F$ composition of two fuzzy binary relations $P \in [0,1]^{X \times Y}$ and $Q \in [0,1]^{Y \times Z}$, is the fuzzy relation given by:*

$$(P \underset{\vee}{@} Q)(x,z) = \bigvee_{y \in Y} F(P(x,y), Q(y,z))$$

for all $x \in X$, $z \in Z$.

Now we present the properties of the previous composition analyzed in [18], in order to show that they verify analogous properties to the $sup - t$ composition [12].

The basic properties of the $sup - F$ composition under standard fuzzy union and intersection, that follow directly from commutativity and monotonicity of the aggregation F, are the following:

$$(\bigcup_{i \in I} P_i) \underset{\vee}{@} Q = \bigcup_{i \in I} (P_i \underset{\vee}{@} Q) \quad (1)$$

$$P \underset{\vee}{@} (\bigcup_{i \in I} Q_i) = \bigcup_{i \in I} (P \underset{\vee}{@} Q_i) \quad (2)$$

$$(\bigcap_{i \in I} P_i) \underset{\vee}{@} Q \subseteq \bigcap_{i \in I} (P_i \underset{\vee}{@} Q) \quad (3)$$

$$P \underset{\vee}{@} (\bigcap_{i \in I} Q_i) \subseteq \bigcap_{i \in I} (P \underset{\vee}{@} Q_i) \quad (4)$$

$$(P \underset{\vee}{@} Q)^{-1} = Q^{-1} \underset{\vee}{@} P^{-1} \quad (5)$$

for any fuzzy relations $P, P_i \in [0,1]^{X \times Y}$ and $Q, Q_i \in [0,1]^{Y \times Z}$, where i takes values in a finite index set I. However, this composition are obviously not commutative.

Remark: The study of the basic properties that remain valid for arbitrary index sets is not included because the aim of this paper is to present the main properties of this composition. However, we are analyzing this topic.

The $sup - F$ composition is also monotonic non-decreasing, that is, for any fuzzy relations $P \in [0,1]^{X \times Y}$, $Q_1, Q_2 \in [0,1]^{Y \times Z}$ and $R \in [0,1]^{Z \times T}$, if $Q_1 \subseteq Q_2$ then

$$P \underset{\vee}{@} Q_1 \subseteq P \underset{\vee}{@} Q_2 \qquad Q_1 \underset{\vee}{@} R \subseteq Q_2 \underset{\vee}{@} R \quad (6)$$

In relation to the associativity of the $sup - F$ composition, we have the following propositions [18]:
1) If the universal sets Y and Z are finite, then $[P \underset{\vee}{@} Q] \underset{\vee}{@} R = P \underset{\vee}{@} [Q \underset{\vee}{@} R]$ for any $P \in [0,1]^{X \times Y}$, $Q \in [0,1]^{Y \times Z}$ and $R \in [0,1]^{Z \times T}$, if and only if the aggregation is associative.
2) If the universal sets Y and Z are not finite, then $[P \underset{\vee}{@} Q] \underset{\vee}{@} R = P \underset{\vee}{@} [Q \underset{\vee}{@} R]$ for any $P \in [0,1]^{X \times Y}$, $Q \in [0,1]^{Y \times Z}$ and $R \in [0,1]^{Z \times T}$, if and only if the aggregation is associative and lower semicontinuous.

As we have said, the aggregation F is associative if and only if is a t-norm or a t-conorm, then t-norms and t-conorms are the only aggregations of Mayor and Torrens that verify these last two propositions. That confirms that the $sup - F$ composition generalizes the $sup - t$ composition and have anologous properties.

3 PSEUDOCOMPLEMENT ASSOCIATED WITH AN AGGREGATION

Weber [21] introduced in 1983 the concept of t-relative pseudocomplement associated with a continuous t-norm, also called residual operator by Fodor and Roubens [10]. This operator have been applied in the resolution of fuzzy relation equations with composition based on t-norms by Miyakoshi and Shimbo in [15] and Di Nola et al. in [7].

In this section we are going to present the pseudocomplement associated with an aggregation lower semicontinuous, as a generalization of the t-relative pseudocomplement of a t-norm. Moreover, we analyze its properties to show that their are analogous to the properties of the t-relative pseudocomplement analyzed by Miyakoshi and Shimbo in [15] and Klir and Yuan in [12].

Definition 3. *Given a lower semicontinuous aggregation F of Mayor and Torrens with $k = F(0,1) = 0$, its pseudocomplement φ is given by*

$$a \varphi b = sup\{x \in [0,1] : F(a,x) \leq b\}$$

for every $a, b \in [0,1]$.

We note that the aggregation operator F must verify $k = F(0,1) = 0$ in order to let the set $\{x \in [0,1] : F(a,x) \leq b\}$ be non-empty, but only the lower semicontinuity of F is needed.

As we have said, this operator generalizes the defined with a continuous t-norm and plays an important role in fuzzy relation equations ([15], [12]). While the aggregation F may be interpreted as logical conjunction, the corresponding operation φ may be interpreted as logical implication. Basic properties of φ are expressed by the following theorems.

Theorem 1. *For any $a, a_1, a_2, b, b_1, b_2, c \in [0,1]$, operation φ verifies:*
(p-1) $c \leq a\varphi b$ iff $F(a,c) \leq b$
(p-2) $(a\varphi b)\varphi b \geq a$
(p-3) $b_1 \leq b_2$ implies $a\varphi b_1 \leq a\varphi b_2$
(p-4) $a_1 \leq a_2$ implies $a_1\varphi b \geq a_2\varphi b$
(p-5) $F(a, a\varphi b) \leq b$
(p-6) $a\varphi F(a,b) \geq b$

Theorem 2. *For any $a, a_i, b, b_i \in [0,1]$, where i takes values from and index set, operation φ verifies:*
(q-1) $(\bigvee_{i\in I} a_i)\varphi b = \bigwedge_{i\in I}(a_i\varphi b)$
(q-2) $(\bigwedge_{i\in I} a_i)\varphi b \geq \bigvee_{i\in I}(a_i\varphi b)$
(q-3) $a\varphi(\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a\varphi b_i)$
(q-4) $a\varphi(\bigvee_{i\in I} b_i) \geq \bigvee_{i\in I}(a\varphi b_i)$

4 INF-φ COMPOSITION OF FUZZY RELATIONS

Given an aggregation operator F of Mayor and Torrens and the associated operator φ, the $inf - \varphi$ composition of two fuzzy relations is given in the next definition.

Definition 4. *The $inf - \varphi$ composition of two fuzzy binary relations $P \in [0,1]^{X\times Y}$ and $Q \in [0,1]^{Y\times Z}$, is the fuzzy relation given by:*

$$(P \underset{\wedge}{\Phi} Q)(x,z) = \bigwedge_{y\in Y}(P(x,y)\varphi Q(y,z))$$

for all $x \in X$, $z \in Z$.

Basic properties of the $inf - \varphi$ composition under standard fuzzy union and intersection, that follow directly from theorem 2, are the following:

$$(\bigcup_{i\in I} P_i)\underset{\wedge}{\Phi} Q = \bigcap_{i\in I}(P_i \underset{\wedge}{\Phi} Q) \tag{7}$$

$$(\bigcap_{i\in I} P_i)\underset{\wedge}{\Phi} Q \supseteq \bigcup_{i\in I}(P_i \underset{\wedge}{\Phi} Q) \tag{8}$$

$$P\underset{\wedge}{\Phi}(\bigcap_{i\in I} Q_i) = \bigcap_{i\in I}(P \underset{\wedge}{\Phi} Q_i) \tag{9}$$

$$P\underset{\wedge}{\Phi}(\bigcup_{i\in I} Q_i) \supseteq \bigcup_{i\in I}(P \underset{\wedge}{\Phi} Q_i) \tag{10}$$

for any fuzzy relations $P, P_i \in [0,1]^{X\times Y}$ and $Q, Q_i \in [0,1]^{Y\times Z}$, where i takes values in a finite index set I.

The previous properties allows us to prove two properties of monotonicity of the $inf - \varphi$ composition: for any fuzzy relations $P \in [0,1]^{X\times Y}$, $Q_1, Q_2 \in [0,1]^{Y\times Z}$ and $R \in [0,1]^{Z\times T}$, if $Q_1 \subseteq Q_2$ then

$$P\underset{\wedge}{\Phi} Q_1 \subseteq P\underset{\wedge}{\Phi} Q_2 \tag{11}$$

$$Q_1\underset{\wedge}{\Phi} R \supseteq Q_2\underset{\wedge}{\Phi} R \tag{12}$$

These properties enable us to prove two theorems which play an important role in the resolution of fuzzy relation equations based on $sup - F$ and $inf - \varphi$ compositions ([7], [12]), since they provide us a procedure to determine the maximum and minimum solutions of these fuzzy relation equations.

Theorem 3. *Let $P \in [0,1]^{X\times Y}$, $Q \in [0,1]^{Y\times Z}$ and $R \in [0,1]^{Z\times T}$ be fuzzy relations. Then,*

$$P\underset{\vee}{@} Q \subseteq R \quad iff \quad Q \subseteq P^{-1}\underset{\wedge}{\Phi} R \tag{13}$$

Theorem 4. *Let $P \in [0,1]^{X\times Y}$, $Q \in [0,1]^{Y\times Z}$ and $R \in [0,1]^{Z\times T}$ be fuzzy relations. Then,*

$$P^{-1}\underset{\vee}{@}(P\underset{\wedge}{\Phi} Q) \subseteq Q \tag{14}$$

$$R \subseteq P\underset{\wedge}{\Phi}(P^{-1}\underset{\vee}{@} R) \tag{15}$$

$$P \subseteq (P\underset{\wedge}{\Phi} Q)\underset{\wedge}{\Phi} Q^{-1} \tag{16}$$

$$R \subseteq (R\underset{\wedge}{\Phi} Q^{-1})\underset{\wedge}{\Phi} Q \tag{17}$$

The previous properties and theorems have been proved by Klir and Yuan in [12] for the $inf - w_t$ composition based on the operator w_t associated with a continuous t-norm t. Therefore, the $inf - \varphi$ composition generalizes the $inf - w_t$ composition, as we wanted to show.

Conclusions

In this paper we have proved that the $sup - F$ and $inf - \varphi$ compositions of fuzzy relations based on the aggregation operator F of Mayor and Torrens and its pseudocomplement φ, respectively, generalize the $sup - t$ and $inf - w_t$ compositions based on the t-norm t and its pseudocomplement w_t, since they have analogous properties. This analysis is the first step to analyze the solutions of fuzzy relation equations based on these compositions.

In this way, our investigation will continue analyzing the fuzzy relation equations with $sup - F$ and $inf - \varphi$ compositions, for any aggregation operator F and its pseudocomplement φ, and also applying these results in fuzzy decision making, because we think that these properties could be interesting for analysis with fuzzy preference relations and consensus of fuzzy opinions.

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