PROBABILITY OF INTUITIONISTIC FUZZY EVENTS AND THEIR APPLICATIONS IN DECISION MAKING

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Summary

A concept of intuitionistic fuzzy events and their probabilities is introduced. The solution is given by an interval and is consistent with the probability measure proposed by Zadeh for fuzzy sets.

The presented considerations seem to be crucial in decision making where imperfection of information is a rule. There are many aspects of information imperfection and among them uncertainty (randomness) and imprecision (fuzziness) are the most important. In this article we assume that imprecission is modelled by intuitionistic fuzzy sets and uncerainty is modelled by probability theory. Both types of information imperfection are discussed and expressed by common formulas.

Keywords: fuzzy set, intuitionistic fuzzy set, fuzzy event, intuitionistic fuzzy event, fuzzy probability, intuitionistic fuzzy probability.

1 INTRODUCTION

Probability theory has been a powerful tool for dealing with uncertainty. For instance, it has been able to handle uncertain events exemplified by the temperature of air tomorrow. Assuming the availability of some past data, one could has made it possible to calculate the probability of, say, the air temperature between 20^o C and 25^o C . However, events have had to be precisely defined.

In practice, however, people often use events which are imprecisely specified, and questions like "what is the probability of a nice weather tomorrow" are considered well valid in human discourse, though they are beyond the scope of classic probability theory.

Since the very beginning of fuzzy sets theory it has been clear that one of key issues would be to be able to devise some formal tools for expressing probabilities of such imprecisely specified events.

And, indeed, it was Zadeh (1968) who first introduced the concept of a fuzzy event, and introduced the concept of its probability by a natural extension of the probability of a nonfuzzy event. In case the probability was specified as a real number from the unit interval. Later on, some extensions have been proposed whose essence was the introduction of a fuzzy probability of a fuzzy event meant as a fuzzy number in the unit interval.

In the mid-1980s Atanassov introduced the concept of an intuitionistic fuzzy set [cf. Atanassov (1986)]. Basically, his idea was that unlike in conventional fuzzy sets in which imprecision is just modelled by the membership degree from [0,1], and for which the nonmembership degree is just automatically the complementation to 1 of the membership degree, in an intuitionistic fuzzy set both the membership and nonmembership degrees are numbers from [0,1], but their sum is not 1. Thus, one can express a well known psychological fact that a human being who expresses the degree of membership of an element in a fuzzy set, very often does not expresses, when asked, the degree of nonmebership as the complementation to 1.

This idea has led to an interesting theory whose point of departure is such a concept of an intuitionistic fuzzy set.

In this article we introduce the concept of an intuitionistic fuzzy event and a probability of an intuitionistic fuzzy event. The proposed probability lies in an interval and is such that when considered intuitionistic fuzzy event becomes a fuzzy event (by making the hesitancy margin equal to zero) the proposed interval is reduced to the probability of a fuzzy event defined by Zadeh.

Next, we replace an intuitionistic fuzzy event by the best corresponding fuzzy event [cf. Szmidt and Kacprzyk 1998b]. In effect the probability of such an event lies in the middle of the previously obtained interval and is consistent with the solution proposed by Gesternkorn and Manko (1990) and Zadeh.

Next, we consider some other properties of the interval in which a probability of an intuitionistic event lies.

2 MAIN DEFINITIONS AND RESULTS

As opposed to a fuzzy set (Zadeh, 1965) in $X = x$, given by:

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \quad (1)$$

where $\mu_{A'} : X \rightarrow [0,1]$ is the membership function of the fuzzy set $A^{'}$, an intuitionistic fuzzy set (Atanassov, 1986) $A \in X$ is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \quad (2)$$

where: $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad (3)$$

and the numbers $\mu_A(x)$, $\nu_A(x) \in [0,1]$, denote the degree of membership and non-membership of x to A respectively.

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (4)$$

the *intuitionistic index* of x in A and it is a hesitancy degree of whether x belongs to A [cf. Atanassov 1986, 1989, 1994a, 1994b]. It is obvious, that $0 \leq \pi_A(x) \leq 1$ *for each* $x \in X$.

Intuitionistic fuzzy sets can be useful in modeling many real situations, and, examples in negotiation processes are given in Szmidt and Kacprzyk (1996a, 1996b, 1996c, 1997a, 1998f, 1998g]).

Zadeh (1968) published the first work dealing with a probability of fuzzy events. By a fuzzy event $A^{'}$ he meant a fuzzy subset of the elementary space $E^{'}$, i.e. $A^{'} \subset E^{'}$ whose membership function $\mu_{A'}(x)$ was Borel measurable.

Definition 1. Let us assign to every element of a fuzzy event $A^{'} \subset E^{'} = \{x_1, \ldots, x_n\}$ its probability of occurrence, i.e. $p(x_1), \ldots, p(x_n)$. Probability of a fuzzy event is equal to

$$p(A^{'}) = \sum_{i=1}^{n} p(x_i)\mu_{A'}(x_i) \quad (5)$$

Definition 2. By an intuitionistic fuzzy event A we will mean an intuitionistic fuzzy subset belonging to the elementary event space E , i.e. $A \subset E$ whose membership function $\mu_A(x)$ and hesitancy margin $\pi_A(x)$ are Borel mesurable.

Definition 3. Let us assign to every element of an intuitionistic fuzzy event $A \subset E = \{x_1, \ldots, x_n\}$ its probability of occurence, i.e. $p(x_1), \ldots, p(x_n)$.

Minimal probability $p_{min}(A)$ of an intuitionistic fuzzy event A is equal to

$$p_{min}(A) = \sum_{i=1}^{n} p(x_i)\mu_A(x_i) \quad (6)$$

Maximal probability $p_{max}(A)$ of an intuitionistic fuzzy event A is equal to

$$p_{max}(A) = p_{min}(A) + \sum_{i=1}^{n} p(x_i)\pi_A(x_i) \quad (7)$$

so probability of an event A is a number from the interval $[p_{min}(A), p_{max}(A)]$, or

$$p(A) \in [\sum_{i=1}^{n} p(x_i)\mu_A(x_i), \sum_{i=1}^{n} p(x_i)\mu_A(x_i) + + \sum_{i=1}^{n} p(x_i)\pi_A(x_i)] \quad (8)$$

Lover boundary of the interval (8) i.e., minimal probability $p_{min}(A)$ (6) gives "sure" or quaranteed probability that an event A will occur.

Upper boundary - maximal probability $p_{max}(A)$ (7) gives the highest possible probability that an event A will occur. The highest probability is achieved when hesitancy margins support occurence of an event A.

The difference between the upper and the lower boundary of the interval (8) i.e., $p_{max}(A) - p_{min}(A)$ is the unsureness of occurence of an intuitionistic event A, i.e.:

$$p_{max}(A) - p_{min}(A) = \sum_{i=1}^{n} p(x_i)\pi_A(x_i) \quad (9)$$

Certainly, the intervals of the estimated probabilities are not totally arbitrary.

- If $p_{max}(A) \leq 1$, then the estimated probabilities are in the interval

$$[0, p_{max}(A)] \quad (10)$$

- If $p_{min}(A) \geq 0$, then the estimated probabilities are in the interval

$$[p_{min}(A), 1] \quad (11)$$

Let us notice that the proposed probability (8) reduces to Zadeh's probability measure (5) in the case when the considered intuitionistic fuzzy event is replaced by ordinary fuzzy set (when assuming that the hesitancy margin is equal to zero). But Atanassov (1994a) proposed a general method of transformation of intuitionistic fuzzy sets into fuzzy sets. It can be done by the following operator

$$\begin{aligned} D_\alpha(A) &= \{< x,\ \mu_A(x) + \alpha\pi_A(x),\ \nu_A(x) + \\ &+ (1-\alpha)\pi_A(x) > | x \in X\} \end{aligned} \tag{12}$$

where $\alpha \in [0, 1]$

Anyway, the definition of $D_\alpha(A)$ does not tell the differences among sets obtained from different parameters α. In Szmidt and Kacprzyk (1998b) we have showed that a good approximation of an intuitionistic fuzzy set via a fuzzy set is given by the operator (12) with α equal to 0.5, i.e.

$$\begin{aligned} D_{0.5}(A) &= \{< x,\ \mu_A(x) + 0.5\pi_A(x),\ \nu_A(x) + \\ &+ 0.5\pi_A(x) > | x \in X\} \end{aligned} \tag{13}$$

Having the above in mind, and taking into account Zadeh's approach (5) we can calculate a probability of such a fuzzy set corresponding to an intuitionistic fuzzy set. The probability of such a fuzzy set lies in the middle of the interval (8), i.e.

$$\begin{aligned} p_{0.5}(A) &= \sum_{i=1}^{n} p_i(x_x)[\mu_A(x_i) + 0.5\pi_A(x_i)] = \\ &= 0.5[p_{min}(A) + p_{max}(A)] \end{aligned} \tag{14}$$

It is worth noticing that (14) can be stated as

$$\begin{aligned} p_{0.5}(A) &= \sum_{i=1}^{n} \frac{p(x_i)}{2} \cdot [2\mu_A(x_i) + \pi_A(x_i)] = \\ &= \sum_{i=1}^{n} \frac{p(x_i)}{2} \cdot [\mu_A(x_i) + 1 - \nu_A(x_i)] \end{aligned} \tag{15}$$

The last expression was proposed in Gerstenkorn and Mako (1990) as a probability of an intuitionistic fuzzy event. It was also demonstrated that $p_{0.5}(A)$ satisfies the properties from the Kolmogorov axiomatic definition of probability as well as other properties characteristic of the classical probability.

Next, when $\pi_A(x_i) = 0$, i.e. for a classical fuzzy set, (14) obviously is reduced to probability of a fuzzy set in the sense of Zadeh's probability.

Let us look at some properties of the border probabilities of A and $\overline{A}$, where $\overline{A}$ denotes the complement of A defined (Atanassov, 1989) the following way

$$\overline{A} = \{< x, \nu_A(x),\ \mu_A(x), > /x \in X\} \tag{16}$$

So we can assign probability that an opposite event $\overline{A}$ will occur (or that an event A will not occur).

Definition 4. Let us assign to every element belonging to the complement of an intuitionistic fuzzy event A i.e., to $\overline{A}$ its probability of occurence, i.e. $p(x_1), \ldots, p(x_n)$.

Minimal probability that an event $\overline{A}$ will occur is equal to

$$p_{min}(\overline{A}) = \sum_{i=1}^{n} p(x_i)\nu_A(x_i) \tag{17}$$

Maximal probability that an event $\overline{A}$ will occur is equal to

$$p_{max}(\overline{A}) = \sum_{i=1}^{n} p(x_i)\nu_A(x_i) + \sum_{i=1}^{n} p(x_i)\pi_A(x_i) \tag{18}$$

Having in mind that $\mu_A(x_i) + \nu_A(x_i) + \pi_A(x_i) = 1$ we have

$$\begin{aligned} p_{min}(\overline{A}) &= \sum_{i=1}^{n} p(x_i)\nu_A(x_i) = \\ &= \sum_{i=1}^{n} p(x_i)[1 - \mu_A(x_i) - \pi_A(x_i)] = \\ &= \sum_{i=1}^{n} [p(x_i) - p(x_i)\mu_A(x_i) - p(x_i)\pi_A(x_i)] = \\ &= 1 - p_{max}(A) \end{aligned} \tag{19}$$

i.e., minimal probability that an opposite event $\overline{A}$ will occur is equal to one minus maximal probability that an event A will occur.

Next,

$$\begin{aligned} p_{max}(\overline{A}) &= \sum_{i=1}^{n} p(x_i)[\nu_A(x_i) + \pi_A(x_i)] = \\ &= \sum_{i=1}^{n} p(x_i)[1 - \mu_A(x_i)] = \\ &= \sum_{i=1}^{n} p(x_i) - \sum_{i=1}^{n} p(x_i)\mu_A(x_i) = \\ &= 1 - p_{min}(A) \end{aligned} \tag{20}$$

what means that maximal probability of an opposite event $\overline{A}$ will occur is equal to one minus minimal probability of an event A.

Next, from (19) we have

$$p_{min}(A) + p_{min}(\overline{A}) = p_{min}(A) + (1 - p_{max}(A)) = = 1 - \sum_{i=1}^{n} p(x_i)\pi_A(x_i) \leq 1 \quad (21)$$

and from (20)

$$p_{max}(A) + p_{max}(\overline{A}) = p_{max}(A) + 1 - p_{min}(A) = = 1 + \sum_{i=1}^{n} p(x_i)\pi_A(x_i) \geq 1 \quad (22)$$

3 FINAL REMARKS

We proposed the probability measure for intuitionistic fuzzy events which is consistent with Zadeh's approach for fuzzy sets.

Next, we investigated some properties of intervals in which the probabilities of an intuitionistic fuzzy event and its complement lie.

The proposed formulas handle two types of imperfection of information - uncertainty (randomness)expressed by probability theory on the one hand, and imprecision (fuzziness) expressed in this article by intuitionistic fuzzy set theory. Both types of imperfection play central role in decision making.

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