An Approach to Interval-Valued R-Implications and Automorphisms

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Abstract— The aim of this work is to introduce an approach for interval-valued R-implications, which satisfy some analogous properties of R-implications. We show that the best interval representation of an R-implication that is obtained from a left continuous t-norm coincides with the interval-valued R-implication obtained from the best interval representation of such t-norm, whenever this is an inclusion monotonic interval function. This provides, under this condition, a nice characterization for the best interval representation of an R-implication, which is also an interval-valued R-implication. We also introduce interval-valued automorphisms as the best interval representations of automorphisms. It is shown that interval automorphisms act on interval R-implications, generating other interval R-implications.

Keywords— Interval-valued fuzzy logic, R-implications, automorphisms, interval representations.

1 Introduction

Interval analysis [1] has been playing an important role in the modeling of the uncertainty and the errors that occur in numerical computations, allowing the development of computational models, methods and tools for the automatic error analysis of numerical algorithms in digital computers, with application in technological and scientific computation.[1]

On the other hand, fuzzy set theory [3] has provided a more generical and complete mathematical model of uncertainty and vagueness, which has been considered as the oldest and most widely reported component of present-day soft computing, helping with the design of flexible information processing systems, with application in, e.g., control systems, decision making, expert systems, pattern recognition. [4]

However, there may be the case that we do not have precise knowledge about the membership function that should be taken in a certain application. These considerations have led to some extensions of fuzzy sets, such us type-2 fuzzy sets [5], which incorporate uncertainty about the membership function into fuzzy set theory. Type-2 fuzzy sets has been largely applied since the works of Jerry Mendel in the 90's [6].

Interval-valued fuzzy sets are a particular case of type-2 fuzzy sets, where the membership degree of each element of the fuzzy set is given by a closed subinterval of the unit interval $[0, 1]$, allowing to deal not only with vagueness (lack of sharp class boundaries), but also with the uncertainty (lack of information), intuitively [7, 8]. Interval-valued fuzzy sets were introduced independently by Zadeh [5] and other authors (e.g., [9, 10, 11, 12]) in the 70's. Among several papers connecting these areas (see, e.g., [13, 14, 15, 16]), we adopted Bedregal and Takahashi's work [17, 18].

Fuzzy implications [19, 20] play an important role in fuzzy logic. They are usually derived from t-norms and t-conorms in several ways, e.g, S-implications, R-implications, QL-implications and D-implications.[2] The importance of fuzzy implications is not only because they are used in representing "If ... then" rules in fuzzy systems, but also because their use in performing inferences in approximate reasoning and fuzzy control. This is the main reason for searching many different models to perform this kind of fuzzy connectives. In particular, R-implications [21], which are based on left continuous t-norms, are used in the modelling of fuzzy rules that satisfy natural properties of implications (see Sect. 4), leading to the residuation property and to an important family of fuzzy logics [22]. On the other hand, the use of automorphisms [19] allows to modify implications preserving their fundamental properties, and then preserving their axiomatic.

The aim of this work is to introduce an approach for interval-valued R-implications as interval generalizations for R-implications that satisfy some analogous properties of R-implications. We show that the best interval representation of an R-implication that is obtained from a left continuous t-norm coincides with the interval-valued R-implication obtained from the best interval representation of such t-norm, whenever this is an inclusion monotonic interval function. Then, it is possible to provide, under this condition, a nice characterization for the best interval representation of an R-implication, which is also an interval-valued R-implication. We also introduce interval-valued automorphisms [16] as the best interval representations of automorphisms, showing that interval automorphisms act on interval R-implications, generating other interval R-implications. Commutative diagrams are used to analyze the relationship between the process for obtaining R-implications from t-norms and the process for obtaining interval-valued R-implications from interval-valued t-norms and those canonical interval constructions. Also, we show that the use of automorphism over R-implications and the use of interval-valued automorphisms over interval-valued

[1] For a survey on applications of Interval Mathematics, see, e.g., [2] and http://www.cs.utep.edu/interval-comp/.

[2] There are also some other methods for the generation of fuzzy implications (see, e.g, [20]).

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R-implications also comute when the interval constructor is applied.

The paper is organized as follows. In Sect. 2, the main concepts related to interval representations are presented. Section 3 considers interval-valued fuzzy t-norms. Fuzzy implications and R-implications, together with their properties, are presented in Sect. 4. Interval-valued fuzzy implications and R-implications are introduced in Sect. 5, and interval-valued automorphism in Sect. 6, as well as the main related results of the paper. Section 6 is the Conclusion and final remarks.

2 Interval Representations

Consider the real unit interval $U = [0,1] \subseteq \mathbb{R}$ and let $\mathbb{U}$ be the set of subintervals of U, that is, $\mathbb{U} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$. The interval set has two projections $l, r : \mathbb{U} \to U$, defined by $l([a,b]) = a$ and $r([a,b]) = b$, respectively. For $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

The partial orders that are considered in this paper are the inclusion relation and the component-wise *Kulisch-Miranker order* (also called *product order*), defined by:

$$\forall X, Y \in \mathbb{U} : X \leq Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}. \qquad (1)$$

An interval $X \in \mathbb{U}$ is said to be an interval representation of a real number α if $\alpha \in X$. Considering two interval representations X and Y of a real number α, X is said to be a better representation of α than Y if $X \subseteq Y$. This notion can be easily extended for tuples of n intervals $\overrightarrow{X} = (X_1, \ldots, X_n)$.

Definition 1 *A function $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an* interval representation *of a function $f : U^n \longrightarrow U$ if, for each $\overrightarrow{X} \in \mathbb{U}^n$ and $\overrightarrow{x} \in \overrightarrow{X}$, $f(\overrightarrow{x}) \in F(\overrightarrow{X})$ [23].*[3]

Definition 2 *Let $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ and $G : \mathbb{U}^n \longrightarrow \mathbb{U}$ be two interval representations of the function $f : U^n \longrightarrow U$. F is a* better interval representation *of f than G, denoted by $G \sqsubseteq F$, if, for each $\overrightarrow{X} \in \mathbb{U}^n$, the inclusion $F(\overrightarrow{X}) \subseteq G(\overrightarrow{X})$ holds.*

Definition 3 *For each real function $f : U^n \longrightarrow U$, the interval function $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$, defined by*

$$\widehat{f}(\overrightarrow{X}) = \left[\inf\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}, \sup\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}\right], \qquad (2)$$

is called the best interval representation *of f [23].*[4]

The interval function $\widehat{f}$ is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. The interval function $\widehat{f}$ returns an interval that is narrower than any other interval representation of f. Thus, $\widehat{f}$ presents the *optimality property* of interval algorithms mentioned by Hickey et al. [24], when it is seen as an algorithm to compute a real function f.

Notice that if f is continuous in the usual sense, then for each $\overrightarrow{X} \in \mathbb{U}^n$, the interval function $\widehat{f}$ applied to $\overrightarrow{X}$ coincides with the image of f when applied to $\overrightarrow{X}$, that is, $\widehat{f}(\overrightarrow{X}) = f(\overrightarrow{X})$, where $f(\overrightarrow{X}) = \{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}$.

In this paper we will take into consideration the following notions of continuity for interval functions:

- *Moore* continuity: It is obtained as an extension of the continuity in the reals, considering the metric $d(X,Y) = \max(\mid \underline{X} - \underline{Y} \mid, \mid \overline{X} - \overline{Y} \mid)$ defined over $\mathbb{U}$.
- *Scott* continuity: The set $\mathbb{U}$ with reverse inclusion order is defined as a continuous domain [25], and a function $f : (\mathbb{U}, \supseteq) \to (\mathbb{U}, \supseteq)$ is Scott continuous if it is monotonic and preserves the least upper bound of directed sets [25].

The main result in [23] can be adapted to our context, considering U^n instead of $\mathbb{R}$, as shown in the following:

Theorem 4 *Let $f : U^n \longrightarrow U$ be a function. The following statements are equivalent: (i) f is continuous; (ii) $\widehat{f}$ is Scott continuous; (iii) $\widehat{f}$ is Moore continuous.*

3 Interval t-norms

A *triangular norm* (t-norm, for short) is a function $T : U^2 \to U$ that is commutative, associative, monotonic and has 1 as neutral element [26]. In the following definition, a natural extension of the t-norm notion for $\mathbb{U}$ is considered, following the approach introduced in [17].

Definition 5 *A function $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is an* interval t-norm *if it is commutative, associative, monotonic with respect to the product and inclusion order and $[1,1]$ is a neutral element.*

An interval t-norm may be considered as an interval representation of a t-norm. This generalization fits with the fuzzy principle, which means that the interval membership degree may be thought as an approximation of the exact degree.

See [18] for the proofs of the next two propositions in this section. The following result shows how an interval t-norm can be constructed from two given t-norms.

Proposition 6 *A function $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-norm if and only if there exist t-norms $\underline{\mathbb{T}}$ and $\overline{\mathbb{T}}$ with $\underline{\mathbb{T}} \leq \overline{\mathbb{T}}$ and*

$$\mathbb{T}(X,Y) = [\underline{\mathbb{T}}(\underline{X}, \underline{Y}), \overline{\mathbb{T}}(\overline{X}, \overline{Y})]. \qquad (3)$$

The next proposition states that the best interval representation of a t-norm is an interval t-norm. It also shows that the interval representation of a t-norm coincides with the interval construction of Prop. 6 when both t-norms are the same.

Proposition 7 *Let T be a t-norm. Then $\widehat{T} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-norm, such that*

$$\widehat{T}(X,Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})]. \qquad (4)$$

4 Fuzzy Implications and R-Implications

Several definitions for fuzzy implication together with related properties have been given (see, e.g., [21, 19, 20]). The unique consensus in these definitions is that the fuzzy implication should present the same behavior of the classical implication for the crisp case. Thus, a function $I : U^2 \longrightarrow U$ is a *fuzzy implication* if it satisfies the minimal boundary conditions:

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

Several reasonable properties may be required for fuzzy implications. The properties considered in this paper are:

I1 : If $y \leq z$ then $I(x,y) \leq I(x,z)$; (2-place monotonicity)

I2 : $I(x, I(y,z)) = I(y, I(x,z))$ (exchange principle);

I3 : $I(x,y) = 1$ if and only if $x \leq y$;

I4 : $\lim_{n\to\infty} I(x, y_n) = I(x, \lim_{n\to\infty} y_n)$ (right-continuity);

I5 : If $x \leq z$ then $I(x,y) \geq I(z,y)$; (1-place antitonicity)

Proposition 8 *[19, Lemma 1 (xi)] If I is a fuzzy implication satisfying* **I1**, **I2** *and* **I3**, *then I also satisfies* **I5**.

[3] Notice that the concept of interval representation is different from interval extension and natural extension. [1, page 21]

[4] Notice that $\widehat{f}$ is the interval hull of the range of f.

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Considering a t-norm T, the equation

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}, \quad (5)$$

defines a fuzzy implication, called *R-implication* or *residuum* of T [21]. R-implications arise from the notion of residuum in Intuitionistic Logic [21] or, equivalently, from the notion of residue in the theory of lattice-ordered semigroups [27]. Observe that an R-implication is well-defined only if the t-norm is left-continuous[5] [19]. This justifies the name "residuum of T", since an R-implication satisfies the residuation condition when the underlying t-norm is left continuous:

$$T(x,z) \le y \text{ if and only if } I_T(x,y) \ge z. \quad (6)$$

Moreover, a t-norm T is left-continuous if and only if it satisfies the residuation condition [28].

The main results relating R-implications and the properties **I1 – I5** are presented in the following.

Theorem 9 *[29, Theorem 1.14] A fuzzy implication $I : U^2 \to U$ is an R-implication with a left-continuous underlying t-norm if and only if I satisfies the properties* **I1–I4**.

Since, by Prop. 8, **I5** follows directly from **I1 – I3**, then R-implications with left-continuous underlying t-norms satisfy the properties **I1 – I5**.

5 Interval-valued Fuzzy R-Implications

The minimal properties required for fuzzy implications can be naturally extended to consider interval fuzzy degrees, by using degenerate intervals. Thus, a function $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an *interval fuzzy implication* if the following conditions hold:

$$\mathbb{I}([1,1],[1,1]) = \mathbb{I}([0,0],[0,0]) = \mathbb{I}([0,0],[1,1]) = [1,1]; \quad (7)$$

$$\mathbb{I}([1,1],[0,0]) = [0,0]. \quad (8)$$

Other properties (see Sect. 4) can be naturally extended:

$\mathbb{I}$**1** : If $Y \le Z$ then $\mathbb{I}(X,Y) \le \mathbb{I}(X,Z)$;

$\mathbb{I}$**2** : $\mathbb{I}(X,\mathbb{I}(Y,Z)) = \mathbb{I}(Y,\mathbb{I}(X,Z))$;

$\mathbb{I}$3**a** : If $\mathbb{I}(X,Y) = [1,1]$ then $X \le Y$;

$\mathbb{I}$3**b** : If $\overline{X} \le \underline{Y}$ then $\mathbb{I}(X,Y) = [1,1]$;

$\mathbb{I}$**4** : $\mathbb{I}_Y(X) = \mathbb{I}(X,Y)$ is (Moore, Scott) continuous;

$\mathbb{I}$**5** : If $X \le Z$ then $\mathbb{I}(X,Y) \ge \mathbb{I}(Z,Y)$,

It is always possible to obtain an interval fuzzy implication from any fuzzy implication canonically. Interval fuzzy implications also meet the optimality property and preserve the same properties satisfied by fuzzy implications. The proof of the following proposition is straightforward.

Proposition 10 *If I is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication. [18]*

The next theorem states that the best interval representation of a fuzzy implication preserves, in some sense, the properties **I1–I5** listed in Sect. 4.

Theorem 11 *Let I be a fuzzy implication satisfying* **I1**. *If I satisfies a property* **Ik**, *for $k = 1,\ldots,5$, then $\widehat{I}$ satisfies* $\mathbb{I}$**k**.

Proof:

$\mathbb{I}$**1**: If I satisfies **I1**, then $\widehat{I}(X,Y) = [\inf\{I(x,\underline{Y})|x \in X\}, \sup\{I(x,\overline{Y})|x \in X\}]$ and $\widehat{I}(X,Z) = [\inf\{I(x,\underline{Z})|x \in X\}, \sup\{I(x,\overline{Z})|x \in X\}]$. Thus, if $Y \le Z$ then, for each $x \in X$, it holds that $I(x,\underline{Y}) \le I(x,\underline{Z})$ and $I(x,\overline{Y}) \le I(x,\overline{Z})$. It follows that $\inf\{I(x,\underline{Y})|x \in X\} \le \inf\{I(x,\underline{Z}) \mid x \in X\}$, $\sup\{I(x,\overline{Y})|x \in X\} \le \sup\{I(x,\overline{Z})|x \in X\}$, and, thus, $\widehat{I}(X,Y) \le \widehat{I}(X,Z)$.

$\mathbb{I}$**2**: If I satisfies **I2**, then, for each $x,y,z \in U$, one has that $I(x,I(y,z)) = I(y,I(x,z))$. Then, since I satisfies **I1**:

$$\begin{aligned}
&\widehat{I}(X,\widehat{I}(Y,Z))\\
&= \widehat{I}(X,[\inf\{I(y,z)) \mid y \in Y, z \in Z\},\\
&\qquad \sup\{I(y,z)) \mid y \in Y, z \in Z\}])\\
&= \widehat{I}(X,[\inf\{I(y,\underline{Z}))|y \in Y,\}, \sup\{I(y,\overline{Z}))|y \in Y\}])\\
&= [\inf\{I(x,\inf\{I(y,\underline{Z}) \mid y \in Y\}) \mid x \in X\},\\
&\quad \sup\{I(x,\sup\{I(y,\overline{Z}) \mid y \in Y\}) \mid x \in X\}]\\
&= [\inf\{I(x,I(y,\underline{Z})) \mid y \in Y, x \in X\},\\
&\qquad \sup\{I(x,I(y,\overline{Z})) \mid y \in Y, x \in X\}]\\
&= [\inf\{I(y,I(x,\underline{Z})) \mid y \in Y, x \in X\},\\
&\qquad \sup\{I(y,I(x,\overline{Z})) \mid y \in Y, x \in X\}]\\
&= \widehat{I}(Y,\widehat{I}(X,Z))
\end{aligned}$$

$\mathbb{I}$3a: One has that if $\widehat{I}(X,Y) = [1,1]$ then $\inf\{I(x,y)|x \in X, y \in Y\} = 1 = \sup\{I(x,y)|x \in X, y \in Y\}$. However, this is only possible if $\{I(x,y)|x \in X, y \in Y\} = \{1\}$, and, thus, $I(x,y) = 1$, for each $x \in X, y \in Y$. Since I satisfies **I3**, then, for each $x \in X, y \in Y$, it is valid that $x \le y$. This is only possible if $\overline{X} \le \underline{Y}$, that is, $X \le Y$.

$\mathbb{I}$3b: If $\overline{X} \le \underline{Y}$ then, for each $x \in X$ and $y \in Y$, it is valid that $x \le y$. So, one has that $\{I(x,y)|x \in X, y \in Y\} = \{1\}$, $\inf\{I(x,y)|x \in X, y \in Y\} = 1 = \sup\{I(x,y)|x \in X, y \in Y\}$. It follows that $\widehat{I}(X,Y) = [1,1]$.

$\mathbb{I}$**4**: For each $x \in X$, let $I_x : U \longrightarrow U$ be defined by $I_x(y) = I(x,y)$. Thus, I is right-continuous if and only if, for each $x \in X$, I_x is continuous. If I_x is continuous then, by Theorem 4, $\widehat{I_x}$ is Scott (Moore) continuous. Thus:

$$\begin{aligned}
&\widehat{I}(X,Y)\\
&= [\inf\{I(x,y)|x \in X, y \in Y\}, \sup\{I(x,y)|x \in X, y \in Y\}]\\
&= [\inf\{\inf\{I(x,y)|y \in Y\} \mid x \in X\},\\
&\qquad \sup\{\sup\{I(x,y)|y \in Y\}|x \in X\}]\\
&= \bigcup_{x \in X} [\inf\{I(x,y)|y \in Y\}, \sup\{I(x,y)|y \in Y\}] = \bigcup_{x \in X} \widehat{I_x}(Y)
\end{aligned}$$

and then, considering that I_x is (topologically) continuous and the union preserves continuity, one concludes that $\widehat{I}$ is also Scott (Moore) continuous.

$\mathbb{I}$**5**: If I satisfies **I5**, then $\widehat{I}(X,Y) = [\inf\{I(\overline{X},y)|y \in Y\}, \sup\{I(\underline{X},y)|y \in Y\}]$ and $\widehat{I}(Z,Y) = [\inf\{I(\overline{Z},y)|y \in Y\}, \sup\{I(\underline{Z},y)|y \in Y\}]$. Thus, if $X \le Z$ then, for each $y \in Y$, it holds that $I(\underline{X},y) \le I(\underline{Z},y)$ and $I(\overline{X},y) \le I(\overline{Z},y)$. It follows that $\inf\{I(\overline{Z},y)|y \in Y\} \le \inf\{I(\overline{X},y) \mid y \in Y\}$ and $\sup\{I(\underline{Z},y)|y \in Y\} \le \inf\{I(\underline{X},y)|y \in Y\}$, and, thus, $\widehat{I}(X,Y) \ge \widehat{I}(Z,Y)$.

□

The next corollary indicates that the best interval representation of a fuzzy implication satisfying **I1–I3** satisfies the properties $\mathbb{I}$**1**–$\mathbb{I}$**3** and $\mathbb{I}$**5**.

Corollary 12 *Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying* **I1**, **I2** *and* **I3**. *Then $\widehat{I}$ satisfies* $\mathbb{I}$**1**–$\mathbb{I}$**3** *and* $\mathbb{I}$**5**.

[5] A t-norm T is said to be left-continuous whenever $\lim_{n\to\infty} T(x_n,y) = T(\lim_{n\to\infty} x_n, y)$. [26]

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Proof: It follows from Prop. 8 and Theorem 11. □

Next proposition provides, for the best interval representation of a fuzzy implication satisfying the properties **I1**, **I2** and **I3**, a more concrete and simpler characterization than Eq. (2).

Proposition 13 *Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying the properties* **I1**, **I2** *and* **I3**. *Then a characterization of $\widehat{I}$ can be obtained as*

$$\widehat{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]. \quad (9)$$

Proof: By Prop. 8, I also satisfies **I5** and, therefore, it holds that $\widehat{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]$ (see also [18]). □

Theorem 14 *Let $\mathbb{I} : \mathbb{U}^2 \to \mathbb{U}$ be an interval fuzzy implication. If $\mathbb{I}$ satisfies $\mathbb{I}$**1** and $\mathbb{I}$**5** and is inclusion monotonic, then $\mathbb{I}$ is representable by the functions $\underline{\mathbb{I}}(x,y) = l(\mathbb{I}([x,x],[y,y]))$ and $\overline{\mathbb{I}} = r(\mathbb{I}([x,x],[y,y]))$, that is, $\mathbb{I}(X,Y) = [\underline{\mathbb{I}}(\overline{X},\underline{Y}), \overline{\mathbb{I}}(\underline{X},\overline{Y})]$.*

Proof: Given $X, Y \in \mathbb{U}$, since $[\overline{X},\overline{X}] \subseteq X$, $[\underline{Y},\underline{Y}] \subseteq Y$, then, by the $\subseteq$-monotonicity of $\mathbb{I}$, we have that $\mathbb{I}([\overline{X},\overline{X}],[\underline{Y},\underline{Y}]) \subseteq \mathbb{I}(X,Y)$, and, thus, $l(\mathbb{I}([\overline{X},\overline{X}],[\underline{Y},\underline{Y}])) \geq l(\mathbb{I}(X,Y))$. On the other hand, since $X \leq [\overline{X},\overline{X}]$ and $[\underline{Y},\underline{Y}] \leq Y$, then, by $\mathbb{I}$**1** and $\mathbb{I}$**5**, it holds that $\mathbb{I}([\overline{X},\overline{X}],[\underline{Y},\underline{Y}]) \leq \mathbb{I}(X,Y)$, and, thus, $l(\mathbb{I}([\overline{X},\overline{X}],[\underline{Y},\underline{Y}])) \leq l(\mathbb{I}(X,Y))$. It follows that $l(\mathbb{I}([\overline{X},\overline{X}],[\underline{Y},\underline{Y}])) = l(\mathbb{I}(X,Y))$. Analogously, since $[\underline{X},\underline{X}] \subseteq X$ and $[\overline{Y},\overline{Y}] \subseteq Y$, then, by the inclusion monotonicity of $\mathbb{I}$, we have that $\mathbb{I}([\underline{X},\underline{X}],[\overline{Y},\overline{Y}]) \subseteq \mathbb{I}(X,Y)$, and, thus, $r(\mathbb{I}([\underline{X},\underline{X}],[\overline{Y},\overline{Y}])) \leq r(\mathbb{I}(X,Y))$. On the other hand, since $[\underline{X},\underline{X}] \leq X$ and $Y \leq [\overline{Y},\overline{Y}]$, then, by $\mathbb{I}$**1** and $\mathbb{I}$**5**, it holds that $\mathbb{I}(X,Y) \leq \mathbb{I}([\underline{X},\underline{X}],[\overline{Y},\overline{Y}])$, and, thus, $r(\mathbb{I}([\underline{X},\underline{X}],[\overline{Y},\overline{Y}])) \geq r(\mathbb{I}(X,Y))$. One has that $r(\mathbb{I}([\underline{X},\underline{X}],[\overline{Y},\overline{Y}])) = r(\mathbb{I}(X,Y))$, and then $\mathbb{I}(X,Y) = [\underline{\mathbb{I}}(\overline{X},\underline{Y}), \overline{\mathbb{I}}(\underline{X},\overline{Y})]$. □

Definition 15 *An interval fuzzy implication $\mathbb{I}$ is an* interval R-implication *if there is an interval t-norm $\mathbb{T}$ such that*

$$\mathbb{I} = \mathbb{I}_{\mathbb{T}}(X,Y) = \sup\{Z \in \mathbb{U} | \mathbb{T}(X,Z) \leq Y\}. \quad (10)$$

Analogously, given an interval fuzzy implication $\mathbb{I}$, one has that $\mathbb{T}_{\mathbb{I}}(X,Y) = \inf\{Z \in \mathbb{U} \mid \mathbb{I}(X,Z) \geq Y\}$.

In Eq. (10), the supremum is determined considering the product order, and, thus, it results from the supremum considering the usual order on the real numbers (the interval endpoints).

The following theorem provides necessary conditions for an interval implication to be an interval R-implication.

Theorem 16 *Let $\mathbb{T}$ be a (Moore, Scott) left-continuous interval t-norm. Then $\mathbb{I}_{\mathbb{T}}$ satisfies $\mathbb{I}$**1**, and $\mathbb{I}$**3**–$\mathbb{I}$**5**.*

Proof: It follows that :

$\mathbb{I}$**1**: If $Y \leq Z$ and $\mathbb{T}(X,Z') \leq Y$ then it holds that $\mathbb{T}(X,Z') \leq Z$. It follows that $\{Z' \in \mathbb{U} \mid \mathbb{T}(X,Z') \leq Y\} \subseteq \{Z' \in \mathbb{U} \mid \mathbb{T}(X,Z') \leq Z\}$, and, thus, $\mathbb{I}_{\mathbb{T}}(X,Y) \leq \mathbb{I}_{\mathbb{T}}(X,Z)$.

$\mathbb{I}$**3**: $\mathbb{I}_{\mathbb{T}}(X,Y) = [1,1] \Leftrightarrow \sup\{Z \in \mathbb{U} | \mathbb{T}(X,Z) \leq Y\} = [1,1] \Leftrightarrow \{Z \in \mathbb{U} | \mathbb{T}(X,Z) \leq Y\} = \mathbb{U} \Leftrightarrow X = \mathbb{T}(X,[1,1]) \leq Y$.

$\mathbb{I}$**4**: It follows from the Moore (Scott) left-continuity of $\mathbb{T}$.

$\mathbb{I}$**5**: If $X \leq X'$ then $\mathbb{T}(X,Z) \leq Y$ implies that $\mathbb{T}(X',Z) \leq Y$. Thus, one has that $\{Z \in \mathbb{U} \mid \mathbb{T}(X',Z) \leq Y\} \subseteq \{Z \in \mathbb{U} \mid \mathbb{T}(X,Z) \leq Y\}$ and then it holds that $\sup\{Z \in \mathbb{U} \mid \mathbb{T}(X',Z) \leq Y\} \leq \sup\{Z \in \mathbb{U} \mid \mathbb{T}(X,Z) \leq Y\}$. It follows that $\mathbb{I}_{\mathbb{T}}(X',Y) \leq \mathbb{I}_{\mathbb{T}}(X,Y)$.

□

In [21, Lemma 5.16], it was proved that $T = T_{I_T}$, where $T_{I_T}(x,y) = \inf\{z \in U \mid I_T(x,z) \geq y\}$. The following two lemmas and proposition shows that this result can be extended for interval R-implications.

Lemma 17 *Let T be a left-continuous t-norm and $\mathbb{T}$ be a Moore left-continuous interval t-norm. If $\widehat{I_T} = \mathbb{I}_{\mathbb{T}}$ then $\mathbb{I}_{\mathbb{T}}([x,x],Z) \geq [y,y]$ if and only if $\mathbb{I}_{\mathbb{T}}([x,x],[\underline{Z},\underline{Z}]) \geq [y,y]$.*

Proof: It follows that: $\mathbb{I}_{\mathbb{T}}([x,x],Z) \geq [y,y] \Leftrightarrow \widehat{I_T}([x,x],Z) \geq [y,y] \Leftrightarrow \{I_T(x,z) \mid z \in Z\} \geq [y,y] \Leftrightarrow I_T(x,\underline{Z}) \geq y \Leftrightarrow \widehat{I_T}([x,x],[\underline{Z},\underline{Z}]) \geq [y,y] \Leftrightarrow \mathbb{I}_{\mathbb{T}}([x,x],[\underline{Z},\underline{Z}]) \geq [y,y]$. □

Lemma 18 *Let T be a left-continuous t-norm and $\mathbb{T}$ be a Moore left-continuous interval t-norm. If $\widehat{I_T} = \mathbb{I}_{\mathbb{T}}$ then it holds that $\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}([x,x],[y,y]) = \mathbb{T}([x,x],[y,y])$.*

Proof: It follows that:

$$\begin{aligned}
&\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}([x,x],[y,y]) \\
&= \inf\{Z \in \mathbb{U} \mid \mathbb{I}_{\mathbb{T}}([x,x],Z) \geq [y,y]\} \\
&= \inf\{[\underline{Z},\underline{Z}] \in \mathbb{U} \mid \mathbb{I}_{\mathbb{T}}([x,x],[\underline{Z},\underline{Z}]) \geq [y,y]\} \text{ by Lemma 17} \\
&= \inf\{[\underline{Z},\underline{Z}] \in \mathbb{U} | \sup\{Z' \in \mathbb{U} | \mathbb{T}([x,x],Z') \leq [\underline{Z},\underline{Z}]\} \geq [y,y]\} \\
&= \inf\{[\underline{Z},\underline{Z}] \in \mathbb{U} \mid \sup\{[\overline{Z'},\overline{Z'}] \in \mathbb{U} \mid \mathbb{T}([x,x],[\overline{Z'},\overline{Z'}]) \\
&\qquad \leq [\underline{Z},\underline{Z}]\} \geq [y,y]\} \text{ by } \subseteq\text{-monotonicity of } \mathbb{T} \\
&= \inf\{[\underline{Z},\underline{Z}] \in \mathbb{U} \mid I_{\underline{\mathbb{T}}}(x,\underline{Z}) \leq y \wedge I_{\overline{\mathbb{T}}}(x,\underline{Z}) \leq y\} \\
&= [T_{I_{\underline{\mathbb{T}}}}(x,y), T_{I_{\overline{\mathbb{T}}}}(x,y)] = [\underline{\mathbb{T}}(x,y), \overline{\mathbb{T}}(x,y)] = \mathbb{T}([x,x],[y,y])
\end{aligned}$$

□

Proposition 19 *Let T be a left-continuous and $\mathbb{T}$ be a Moore left-continuous interval t-norms. If $\widehat{I_T} = \mathbb{I}_{\mathbb{T}}$ then $\mathbb{T}_{\mathbb{I}_{\mathbb{T}}} = \mathbb{T}$.*

Proof: It follows that:

$$\begin{aligned}
&\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}(X,Y) \\
&= [\underline{\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}}(\underline{X},\underline{Y}), \overline{\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}}(\overline{X},\overline{Y})] \\
&= [l(\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}([\underline{X},\underline{X}],[\underline{Y},\underline{Y}]), r(\mathbb{T}_{\mathbb{I}_{\mathbb{T}}}([\overline{X},\overline{X}],[\overline{Y},\overline{Y}]) \\
&= [l(\mathbb{T}([\underline{X},\underline{X}],[\underline{Y},\underline{Y}])), r(\mathbb{T}([\overline{X},\overline{X}],[\overline{Y},\overline{Y}]))] \text{ by Lemma 18} \\
&= [\underline{\mathbb{T}}([\underline{X},\underline{X}],[\underline{Y},\underline{Y}]), \overline{\mathbb{T}}([\overline{X},\overline{X}],[\overline{Y},\overline{Y}])] = \mathbb{T}(X,Y)
\end{aligned}$$

□

Proposition 20 *Let T be a left-continuous t-norm and $\mathbb{T}$ be a Moore left-continuous interval t-norm. If $\widehat{I_T} = \mathbb{I}_{\mathbb{T}}$ then $\mathbb{T}$ is an interval representation of T.*

Proof: It holds that

$$\begin{aligned}
&\mathbb{T}([x,x],[y,y]) \\
&= \mathbb{T}_{\mathbb{I}_{\mathbb{T}}}([x,x],[y,y]) \text{by Prop. 19} \\
&= \inf\{Z \in \mathbb{U} \mid \mathbb{I}_{\mathbb{T}}([x,x],Z) \geq [y,y]\} \\
&= \inf\{Z \in \mathbb{U} \mid \widehat{I_T}([x,x],Z) \geq [y,y]\} \text{ by hypothesis} \\
&= \inf\{Z \in \mathbb{U} \mid I_T(x,z) \geq y, \forall z \in Z\} \\
&= \inf\{Z \in \mathbb{U} \mid I_T(x,\underline{Z}) \geq y\} \text{ by } \mathbf{I1} \text{ and Theorem 9} \\
&= \inf\{[\underline{Z},\underline{Z}] \in \mathbb{U} \mid I_T(x,\underline{Z}) \geq y\} \\
&= [\inf\{\underline{Z} \in U \mid I_T(x,\underline{Z}) \geq y\}, \inf\{\underline{Z} \in U \mid I_T(x,\underline{Z}) \geq y\}] \\
&= [T_{I_T}(x,y), T_{I_T}(x,y)] = [T(x,y), T(x,y)].
\end{aligned}$$

By the $\subseteq$-monotonicity of $\mathbb{T}$, we have that $\mathbb{T}$ represents T. □

Theorem 21 *If T is a left continuous t-norm, then $\mathbb{I}_{\widehat{T}} \subseteq \widehat{I_T}$.*

Proof: It follows that:

$$\begin{aligned}
&\mathbb{I}_{\widehat{T}}(X,Y) \\
&= \sup\{Z \in \mathbb{U} | \widehat{T}(X,Z) \leq Y\} \text{ by Eq. (10)} \\
&= \sup\{[\underline{Z},\overline{Z}] \in \mathbb{U} | [T(\underline{X},\underline{Z}), T(\overline{X},\overline{Z})] \leq [\underline{Y},\overline{Y}]\} \text{ by Eq. (4)} \\
&= \sup\{[\underline{Z},\overline{Z}] \in \mathbb{U} | T(\underline{X},\underline{Z}) \leq \underline{Y} \wedge T(\overline{X},\overline{Z}) \leq \overline{Y}\} \text{ by Eq. (1)} \\
&= [\min\{\sup\{\underline{Z} \in U | T(\underline{X},\underline{Z}) \leq \underline{Y}\}, \\
&\quad \sup\{\overline{Z} \in U | T(\overline{X},\overline{Z}) \leq \overline{Y}\}\}, \sup\{\overline{Z} \in U | T(\overline{X},\overline{Z}) \leq \overline{Y}\}] \\
&= [\min\{I_T(\underline{X},\underline{Y}), I_T(\overline{X},\overline{Y})\}, I_T(\overline{X},\overline{Y})] \text{ by Eq. (5)} \\
&\subseteq [I_T(\overline{X},\underline{Y}), I_T(\underline{X},\overline{Y})] = \widehat{I_T}(X,Y) \text{ by Theorem 9 and Eq. (9)}
\end{aligned}$$

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□

The following results states a sufficient condition to guarantee the desired equivalence $\mathbb{I}_{\widehat{T}} = \widehat{I_T}$.

Lemma 22 *Let T be left-continuous t-norm. Then, it holds that $\underline{\mathbb{I}_{\widehat{T}}} = I_T = \overline{\mathbb{I}_{\widehat{T}}}$.*

Proof: It follows that

$$\begin{aligned} \underline{\mathbb{I}_{\widehat{T}}}(x,y) &= l(\mathbb{I}_{\widehat{T}}([x,x],[y,y])) \\ &= l(\sup\{Z \in \mathbb{U} \mid \widehat{T}([x,x],Z) \leq [y,y]\}) \\ &= l(\sup\{[\overline{Z},\overline{Z}] \in \mathbb{U} \mid \widehat{T}([x,x],[\overline{Z},\overline{Z}]) \leq [y,y]\}) \\ &\qquad\qquad \text{by } \leq \text{ and } \subseteq\text{-monotonicity} \\ &= \sup\{\overline{Z} \in U \mid T(x,\overline{Z}) \leq y\} = I_T(x,y). \end{aligned}$$

The proof of the other equality is analogous. □

Proposition 23 *If T is left-continuous t-norm then $\widehat{T}$ is a Moore left-continuous interval t-norm.*

Proof: It is analogous to the proof of Theorem 11 ($\mathbb{I}$**4**). □

Theorem 24 *Let T be a left-continuous t-norm. If $\mathbb{I}_{\widehat{T}}$ is $\subseteq$-monotonic then*

$$\mathbb{I}_{\widehat{T}} = \widehat{I_T}. \tag{11}$$

Proof: By Theorem 9 and Prop. 8, I_T satisfies **I1** and **I5** and $\widehat{I_T}(X,Y) = [\inf\{I_T(x,y) \mid x \in X \wedge y \in Y\}, \sup\{I_T(x,y) \mid x \in X \wedge y \in Y\}] = [I_T(\overline{X},\underline{Y}), I_T(\underline{X},\overline{Y})]$. On the other hand, since $\mathbb{I}_{\widehat{T}}$ is $\subseteq$-monotonic, T is (Moore, Scott) left-continuous (by Prop. 23), and, by Theorem 16, it satisfies $\mathbb{I}$**1** and $\mathbb{I}$**5**. By Theorem 14, it follows that $\mathbb{I}_{\widehat{T}}(X,Y) = [\underline{\mathbb{I}_{\widehat{T}}}(\overline{X},\underline{Y}), \overline{\mathbb{I}_{\widehat{T}}}(\underline{X},\overline{Y})]$. Then, by Lemma 22, one has that $\mathbb{I}_{\widehat{T}}(X,Y) = [I_T(\overline{X},\underline{Y}), I_T(\underline{X},\overline{Y})]$. Therefore, it holds that $\mathbb{I}_{\widehat{T}}(X,Y) = \widehat{I_T}(X,Y)$. □

The above results state the commutativity of the diagram of Fig. 1, where $\mathcal{C}(T)$ ($\mathcal{C}(\mathbb{T})$) is the class of left-continuous (interval) t-norms and $\mathcal{C}(I)$ ($\mathcal{C}(\mathbb{I})$) is the class of ($\subseteq$-monotonic interval) R-implications.

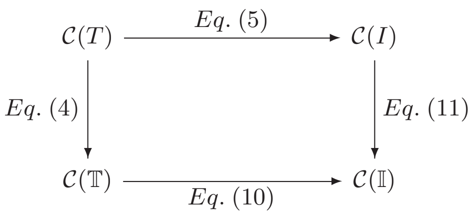

Figure 1: Commutative diagram relating R-implications with interval R-implications

6 Interval-valued Automorphisms

Definition 25 *[30] $\rho : U \longrightarrow U$ is an* automorphism *if it is bijective and monotonic.*[6] *The* action *of an automorphism ρ on a function $f : U^2 \to U$, denoted by f^ρ, is defined as $f^\rho(x,y) = \rho^{-1}(f(\rho(x),\rho(y)))$.*

As it is well known, if T is a t-norm then T^ρ is also a t-norm, and if I is an R-implication then I^ρ is also an R-implication.

$\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an *interval automorphism* if it is bijective and monotonic with respect to the product order [16]. The set of interval automorphisms is denoted by $Aut(\mathbb{U})$.

The next theorem shows that each interval automorphism can be constructed from an automorphism.

[6]In [19], $\rho : U \longrightarrow U$ is an automorphism if it is a continuous and strictly increasing function such that $\rho(0) = 0$ and $\rho(1) = 1$.

Theorem 26 *[16, Theorems 2 and 3] $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval automorphism if and only if there exists an automorphism $\rho : U \longrightarrow U$ such that $\varrho(X) = [\rho(\underline{X}), \rho(\overline{X})]$.*

Clearly, if $\rho : U \to U$ is an automorphism then $\widehat{\rho}$ can be obtained as $\widehat{\rho}(X) = [\rho(\underline{X}), \rho(\overline{X})]$. Therefore, interval automorphisms are the best interval representations of automorphisms.

Remark 1 *Let ϱ be an interval automorphism. Then: (i) ϱ^{-1} is an interval automorphism; (ii) $X \leq Y \Leftrightarrow \varrho(X) \leq \varrho(Y)$.*

Definition 27 *The* action *of an interval automorphism ϱ on an interval function $\mathbb{F}:\mathbb{U}^2 \longrightarrow \mathbb{U}$, denoted by $\mathbb{F}^\varrho$, is defined as*

$$\mathbb{F}^\varrho(X,Y) = \varrho^{-1}(\mathbb{F}(\varrho(X),\varrho(Y))). \tag{12}$$

In the following, we show how interval automorphisms act on interval t-norms and interval R-implications.

Proposition 28 *[17, Theorem 6.1] Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{T} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval t-norm. Then the mapping $\mathbb{T}^\varrho : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an interval t-norm.*

Theorem 29 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{T} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval t-norm. Then the mapping $\mathbb{I}^\varrho_{\mathbb{T}} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is defined by $\mathbb{I}^\varrho_{\mathbb{T}}(X,Y) = \mathbb{I}_{\mathbb{T}^\varrho}(X,Y)$.*

Proof: Since ϱ is bijective and ϱ is monotonic, it follows that:

$$\begin{aligned} &\mathbb{I}^\varrho_{\mathbb{T}}(X,Y) \\ &= \varrho^{-1}(\mathbb{I}_{\mathbb{T}}(\varrho(X),\varrho(Y))) \text{ by Eq. (12)} \\ &= \varrho^{-1}(\sup\{Z \in \mathbb{U} | \mathbb{T}(\varrho(X),Z) \leq \varrho(Y)\}) \text{ by Eq. (10)} \\ &= \varrho^{-1}(\sup\{\varrho(Z') \in \mathbb{U} | \mathbb{T}(\varrho(X),\varrho(Z')) \leq \varrho(Y)\}) \\ &= \sup\{\varrho^{-1}(\varrho(Z')) \in \mathbb{U} | \mathbb{T}(\varrho(X),\varrho(Z')) \leq \varrho(Y)\} \\ &= \sup\{Z' \in \mathbb{U} | \varrho^{-1}(\mathbb{T}(\varrho(X),\varrho(Z'))) \leq \varrho^{-1}(\varrho(Y))\} \text{ by Rem. 1} \\ &= \sup\{Z' \in \mathbb{U} | \varrho^{-1}(\mathbb{T}(\varrho(X),\varrho(Z'))) \leq Y\} \text{ by Rem. 1} \\ &= (\sup\{Z \in \mathbb{U} | \mathbb{T}^\varrho(X,Z) \leq Y\}) \text{ by Eq. (12)} \\ &= \mathbb{I}_{\mathbb{T}^\varrho}(X,Y) \text{ by Eq. (10)}. \end{aligned}$$

□

Corollary 30 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{T} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval t-norm. Then the mapping $\mathbb{I}^\varrho_{\mathbb{T}} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an interval R-implication.*

Theorem 29 and Cor. 30 state that to apply an interval automorphism to an interval R-implication is the same that to apply it to an interval t-norm, and then to obtain an interval R-implication. Whenever an interval R-implication is submitted to an interval automorphism, a new interval R-implication is generated, which means that interval automorphisms may be applied in order to generate new interval R-implications. So, the commutative diagram pictured in Fig. 2 holds.

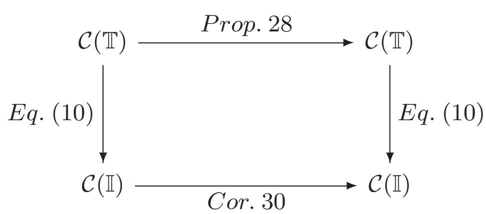

Figure 2: Commutative diagram relating interval t-norms, interval R-implications and interval automorphisms

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7 Conclusion and Final Remarks

The results presented in this paper extend our previous work (e.g., [17, 31]). We show that interval R-implications satisfy some analogous properties of R-implications. The best interval representation of an R-implication that is obtained from a left continuous t-norm coincides with the interval-valued R-implication obtained from the best interval representation of such t-norm, whenever this is a $\subseteq$-monotonic interval function. This provided, under this condition, a nice characterization for the best interval representation of an R-implication. Interval automorphisms are presented as best interval representations of automorphisms, showing that interval automorphisms act on interval R-implications, generating other interval R-implications.

Future work will consider the analysis of other important properties of interval-valued R-implications, in other to obtain a stronger relation between R-implications and interval-valued R-implications, establishing under which conditions interval R-implications are representable in the sense of [32].

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