On the structure of the k-aditive fuzzy measures

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Abstract— The family of k-additive measures has been introduced as a midterm between probabilities and general fuzzy measures and finds a wide number of applications in practice. However, its structure is different from other families of fuzzy measures and is certainly more complex (for instance, its vertices are not always $\{0, 1\}$-valued), so it has not been yet fully studied.

In this paper we present some results concerning the extreme points of the k-additive fuzzy measures. We give a characterization of these vertices as well as an algorithm to compute them. We show some examples of the results of this algorithm and provide lower bounds on the number of vertices of the $n-1$-additive measures, proving that it grows much faster than the number of vertices of the general fuzzy measures. This suggests that k-additive measures might not be a good choice in modeling certain decision problems when the value of k is high but not equal to n.

Keywords— Fuzzy measures, k-additive measures, polytope vertices.

1 Introduction

Fuzzy measures (also known as capacities or non-additive measures) are a generalization of probability distributions. More concretely, they are measures in which the additivity axiom has been relaxed to a monotonicity condition. This extension is needed in many practical situations, in which additivity is too restrictive. Fuzzy measures have proved themselves to be a powerful tool in many different fields, as Decision Theory ([1], [2]), Game Theory ([3], [4]), and many others (see, for example, [5],[6]).

However, despite all these advantages, the practical application of fuzzy measures is limited by the increased complexity of the measure. If we have a finite space of cardinality n, only $n-1$ values are needed in order to completely determine a probability, while $2^n - 2$ coefficients are needed to define a fuzzy measure on the same referential. This exponential growth is the actual *Achilles' heel* of fuzzy measures and translates in big complexity when identifying the fuzzy measure modelling a situation.

With the aim of reducing this complexity several of subfamilies have been defined. In these families some extra restrictions are added in order to decrease the number of coefficients but trying to keep the modelling capabilities of the measures. Examples of subfamilies include the λ-measures [7], the k-intolerant measures [8, 9], the p-symmetric measures [10], the decomposable measures [11], etc. In this paper we will focus on the complexity of k-additive measures (see Definition 9 below) introduced by Grabisch in [4].

Consider a situation that can be modeled by a k-additive measure (possibly through Choquet integral [12]); to this extent, an axiomatic characterization of such a model can been found in [13]. Next step is the identification of the measure. Suppose that sample information is available. Using Decision Theory terminology, this information consists in a collection of objects whose overall score and valuation on each criterium are known. We assume that this information is numerical; otherwise, we should previously apply a tool to transform ordinal into numerical data, such as MACBETH [14] or TOMASO [15]. The desired measure (that might not be unique) is the one that best fits the data.

If the considered proximity criterion is the squared error, different techniques exist to solve the problem [16, 17], leading to a problem whose complexity is very high. In a previous work [18], we have proposed a learning method based on genetic algorithms [19]. In this method, the crossover operator is the convex combination (as it will become clearer below, the set of fuzzy measures being at most k-additive forms a convex polytope). This operator is simple and natural and allows to decrease the computational cost. However, it has the drawback that the search regions are embedded one in other. This can be overcome with a suitable mutation operator, but even in that case, the search region is determined by the initial population. Thus, this initial population must be wide enough to include the solution and, obviously, the best option is to select the set of vertices of the polytope.

In this paper we study this set of vertices for the family of k-additive fuzzy measures. The results in the paper seem to mean that the number of k-additive vertices is even greater than the number of vertices for general fuzzy measures. The rest of the paper is organized as follows. In Section 2 we briefly introduce definitions and results that will be used throughout the paper. In Section 3 we provide an algorithm to determine the vertices of the polytope of k-additive (at most) measures from those of the general fuzzy measures and show some examples of the results of the computation of the algorithm. Section 4 is devoted to the study of lower bounds on the number of vertices when $k = n-1$. Finally, in Section 5 some conclusions are drawn and ideas for future work are presented.

2 Basic concepts and previous results

Consider a finite referential set $X = \{1, ..., n\}$ of n elements or *criteria*. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. Subsets of X are denoted $A, B, ...$ and also by $A_1, A_2, ...$

Definition 1 *[12, 20, 21] A* **fuzzy measure**, **non-additive measure** *or* **capacity** *over X is a function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying*

- $\mu(\emptyset) = 0$, $\mu(X) = 1$ *(boundary conditions).*

- $\forall A, B \in \mathcal{P}(X)$, *if* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$ *(monotonicity).*

We will denote the set of all fuzzy measures over X by $\mathcal{FM}(X)$. Remark that $\mathcal{FM}(X)$ is a bounded convex polyhedron in $\mathbb{R}^{2^n-2}$ (or $\mathbb{R}^{2^n}$ if we include coordinates for X and $\emptyset$).

We can define a partial order on $\mathcal{FM}(X)$ in the following way:

Definition 2 *Let μ_1 and μ_2 be two fuzzy measures. Then we say that $\mu_1 \leq \mu_2$ if for every $A \subseteq X$, it holds $\mu_1(A) \leq \mu_2(A)$.*

A special class of fuzzy measures is the set of $\{0,1\}$-valued measures.

Definition 3 *A fuzzy measure is* $\{0,1\}$**-valued** *if it only takes values 0 and 1.*

Notice that for a $\{0,1\}$-valued measure μ, there are some subsets A satisfying the following conditions:

$$\begin{array}{ll} \mu(A) = 1, & \\ \mu(B) = 1, & \forall B \supseteq A, \\ \mu(C) = 0, & \forall C \subset A. \end{array}$$

We will call any subset satisfying these conditions a **minimal subset** for μ. Note that a $\{0,1\}$-valued measure is completely defined by its minimal subsets. Let us denote by $\mu_{\mathbf{C}}$ the fuzzy measure whose collection of minimal sets is **C**.

An important instance of $\{0,1\}$-valued measures are the so-called unanimity games:

Definition 4 *A* **unanimity game** *over $A \subseteq X$, $A \neq \emptyset$ is a fuzzy measure defined by*

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

For $\emptyset$, we define the unanimity game by

$$u_\emptyset(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

Definition 5 *Let μ be a fuzzy measure over X; we define the* **dual measure of** *μ as the fuzzy measure $\overline{\mu}$ given by $\overline{\mu}(A) = 1 - \mu(A^c)$.*

In order to define the *extreme points* or *vertices* of a family of fuzzy measures we need to define the convex combination of measures.

Definition 6 *Given $\lambda \in [0,1]$ and two fuzzy measures μ_1 and μ_2 the fuzzy measure*

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2$$

is said to be a **convex combination** *of μ_1 and μ_2. A subset of fuzzy measures $\mathcal{F} \subseteq \mathcal{FM}(X)$ is* **convex** *if it contains every convex combination of its members.*

Definition 7 *Given a convex subset of fuzzy measures, $\mathcal{F} \subseteq \mathcal{FM}(X)$, we say that $\mu \in \mathcal{F}$ is a* **vertex** *or* **extreme point** *of $\mathcal{F}$ if μ can not be written as a convex combination of two measures $\mu_1, \mu_2 \in \mathcal{F}$ both different from μ.*

Obviously, the set of vertices completely determines the convex polytope $\mathcal{FM}(X)$, as any point in it can be written as a convex combination of the vertices. The vertices of $\mathcal{FM}(X)$ are given in the following result.

Proposition 1 *[22] The set of $\{0,1\}$-valued measures constitutes the set of vertices of $\mathcal{FM}(X)$.*

A similar result (see [23]) can be proved for the family of p-symmetric measures [10]. However, it does not hold [23] for the family of measures that we study in this paper: the k-additive measures, which are defined through the concept of Möbius transform.

Definition 8 *[6, 24] Let μ be a set function (not necessarily a fuzzy measure) on X. The* **Möbius transform (or inverse)** *of μ is another set function on X defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \; \forall A \subseteq X. \tag{1}$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [25]:

$$\mu(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

The value $m(A)$ represents the strength of the subset A in any superset which it appears. Remark that the Möbius transform can attain negative values; when it is a non-negative function, it corresponds to the *basic probability mass assignment* in Dempster-Shafer theory of evidence [26].

In order to determine a fuzzy measure, $2^n - 2$ values are necessary. The number of coefficients grows exponentially with n and so does the complexity of the problem of identifying the fuzzy measure. This drawback reduces considerably the practical use of fuzzy measures. Thus, some subfamilies of fuzzy measures have been defined in an attempt to reduce complexity, prominently k-additive measures [4].

Definition 9 *[4] A fuzzy measure μ is said to be* k**-order additive** *or* k**-additive** *if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A with exactly k elements such that $m(A) \neq 0$.*

In this sense, a probability measure is just a 1-additive measure. Thus, k-additive measures generalize probability measures, that are very restrictive in many situations. They fill the gap between probability measures and general fuzzy measures. For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i}.$$

More about k-additive measures can be found e.g. in [2]. We will denote the set of all k'-additive measures on X with $k' \leq k$ by $\mathcal{FM}^k(X)$; we will use the fact that $\mathcal{FM}^k(X)$ is a convex polyhedron (the proof is straightforward considering the Möbius transform). Specially appealing is the 2-additive case, that provides a generalization of probability allowing interactions while keeping a reduced complexity.

In this paper we will be studying some properties of the extreme points of the k-additive measures. The following results are known:

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Theorem 1 *[23] There are vertices of the set $\mathcal{FM}^k(X), k > 2$, that are not $\{0,1\}$-valued measures.*

Proposition 2 *[23] The set of extreme points of $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$) are the $\{0,1\}$-valued measures that are in $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$).*

We will also need some results relating the group of isometries of $\mathcal{FM}(X)$ in next section.

Definition 10 *Let $\mathcal{F}$ be a family of fuzzy measures. A surjective function $f : \mathcal{F} \to \mathcal{F}$ is an* **isometry** *if*

$$d(\mu_1, \mu_2) = d(f(\mu_1), f(\mu_2)), \forall \mu_1, \mu_2 \in \mathcal{F}.$$

Remark that an isometry is a bijective mapping on $\mathcal{F}$ and that these isometries form a group under composition of functions. It can be seen [27] that isometries also map vertices into vertices. Let us denote by $\mathcal{G}(\mathcal{F})$ the group of isometries of $\mathcal{F}$.

In [27] we have determined the group of isometries of general fuzzy measures (a result which was later generalized on [28]). We need some previous definitions.

Definition 11 *Consider $\sigma : X \to X$ a permutation on X. We define the* **symmetry induced by** *σ, denoted S_σ, the transformation on $\mathcal{FM}(X)$ such that for any $\mu \in \mathcal{FM}(X)$, the fuzzy measure $S_\sigma(\mu)$ is defined by*

$$S_\sigma(\mu)(A) = \mu(\sigma(A)), \forall A \subseteq X.$$

Definition 12 *We define the* **dual transformation***, denoted D, the transformation on $\mathcal{FM}(X)$ given by*

$$\begin{array}{cccc} D: & \mathcal{FM} & \to & \mathcal{FM} \\ & \mu & \mapsto & \overline{\mu} \end{array}.$$

In most cases, the group of isometries of the general fuzzy measures is generated by symmetries and the dual transformation.

Theorem 2 *If $|X| > 2$, the group $\mathcal{G}(\mathcal{FM}(X))$ is given by symmetries and compositions of symmetries with the dual application.*

In fact, it can be seen that $\mathcal{G}(\mathcal{FM}(X))$ is the semi-direct product of the group of symmetries with the cyclic group or order 2 generated by the dual transformation, i.e. the group of isometries is given by a composition of symmetries composed with either the dual application or the identity map.

Theorem 3 *If $n = 2$, the group $\mathcal{G}(\mathcal{FM}(X))$ is isomorphic to the dihedral group D_4 (the group of isometries of the square).*

Finally, we define the concept of adjacency of extreme points of $\mathcal{FM}(X)$, which will be central in our study of the vertices of $\mathcal{FM}^k(X)$.

Definition 13 *Two vertices μ_1 and μ_2 of $\mathcal{FM}(X)$ are adjacent if the convex combination (their midpoint)*

$$\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$$

can not be written as a convex combination of extreme points in any other way.

The following results have been proved in [29] and later generalized in [30].

Lemma 1 *If μ_1 and μ_2 are adjacent vertices then $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$.*

However, this condition is not sufficient. The characterization of adjacency in $\mathcal{FM}(X)$ is based on the concept of $\mathcal{C}-$decomposability, that is given in next definition.

Definition 14 *Let $\mathbf{C}$ be a collection of subsets of X, μ an extreme point and let $A_1, \ldots, A_m$ be all its minimal sets. We say that μ is* **C-decomposable** *if there exists a partition of $\{A_1, \ldots, A_m\}$ in two non-empty subsets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \not\subseteq \mathbf{C}$ and $\mathbf{B} \not\subseteq \mathbf{C}$ and if $A \in \mathbf{A}$ and $B \in \mathbf{B}$ then there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$.*

The following theorem characterizes adjacency on $\mathcal{FM}(X)$.

Theorem 4 *Let $\mu, \mu_\mathcal{C}$ be two vertices of $\mathcal{FM}(X)$ such that $\mu \geq \mu_\mathcal{C}$. Then μ and $\mu_\mathbf{C}$ are adjacent if and only if μ is not $\mathbf{C}$-decomposable.*

3 Generating the vertices of the k-additive measures

Consider the set $\mathcal{FM}(X)$. From a geometrical point of view, $\mathcal{FM}^k(X)$ is a subpolytope of $\mathcal{FM}(X)$ with additional restrictions; namely, these restrictions are $m(A) = 0, \forall |A| > k$.

Let $|A| > k$ and consider the restriction $m(A) = 0$. Then, it can be proved that the set of vertices of the polytope of fuzzy measures with this new condition consists in (see [31]):

1. Those $\{0,1\}$-valued measures for which it is $m(A) = 0$ and
2. the cut points of the hyper-plane $m(A) = 0$ with the edges of the polytope $\mathcal{FM}(X)$.

If we repeat the procedure imposing restrictions for every set A with size bigger than k we eventually obtain all the vertices of the polytope $\mathcal{FM}^k(X)$. Thus, we obtain a characterization of the extreme points of the k-additive measures (and of all intermediate polytopes). If $\mathbf{A}$ is a collection of subsets of X we denote by $\mathcal{FM}_\mathbf{A}(X)$ the set of fuzzy measures whose Möbius transform is zero on every set belonging to $\mathbf{A}$. The following theorem, well-known in the theory of polytopes, provides the new vertices when adding a new constraint.

Theorem 5 *Let $\mathbf{A}$ be a collection of subsets of X (eventually empty), B a subset of X not in $\mathbf{A}$ and μ a fuzzy measure of $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ which is not a vertex of $\mathcal{FM}_\mathbf{A}(X)$. Then μ is a vertex of $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ if and only if there exist μ_1 and μ_2 two adjacent vertices of $\mathcal{FM}_\mathbf{A}(X)$ not in $\mathcal{FM}_{\mathbf{A}\cup\{B\}}(X)$ and $\lambda \in (0,1)$ such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$.*

As a consequence of this result, for the case of $(n-1)$-additive measures we have the following:

Theorem 6 *Let μ be a $(n-1)$-additive measure which is not $\{0,1\}$-valued. Then μ is a vertex of $\mathcal{FM}^{n-1}(X)$ if and only*

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2,$$

where $\lambda \in (0,1)$ and μ_1, μ_2 are two adjacent vertices of $\mathcal{FM}(X)$ not in $\mathcal{FM}^{n-1}(X)$.

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In this particular case we can obtain all vertices of $\mathcal{FM}^{n-1}(X)$ just from adjacency relationships in $\mathcal{FM}(X)$. However, this is not possible in general; for other values of k there exist vertices not $\{0,1\}$-valued which cannot be expressed as a convex combination of exactly two $\{0,1\}$-valued vertices, as the following example shows.

Example 1 *Consider $|X| = 5$. Let μ_1 be the $\{0,1\}$-valued measure whose minimal sets are the singletons $\{1\},\ldots,\{5\}$ and μ_2 the fuzzy measure whose minimal sets are the sets of size 4. Also, let μ_3 be the fuzzy measure with minimal sets $\{(1,2),(1,3),(2,4),(3,5),(4,5)\}$.*

From Theorem 4 it follows that μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}(X)$. Analogously, μ_2 and μ_3 are also adjacent vertices.

It is easy to check that

$$\frac{4}{5}\mu_1 + \frac{1}{5}\mu_2, \frac{4}{5}\mu_3 + \frac{1}{5}\mu_2 \in \mathcal{FM}^4(X). \tag{3}$$

Thus, these measures are extreme points of $\mathcal{FM}^4(X)$ and are not $\{0,1\}$-valued. Let us denote them by μ and μ' respectively. It can be checked that these measures are adjacent in $\mathcal{FM}^4(X)$. In fact, it easy to verify that their midpoint satisfies with equality exactly $2^n - 3$ linearly independent restrictions of those defining the polytope. Moreover,

$$\frac{1}{4}\mu + \frac{3}{4}\mu' \in \mathcal{FM}^3(X). \tag{4}$$

Thus, we have obtained a vertex of the 3-additive measures, which requires three vertices of $\mathcal{FM}(X)$ in order to obtain a convex combination which generates it.

As an application of Theorem 5 we have computed the vertices of the polytopes $\mathcal{FM}^k(X)$ for some values of k and n. The following tables show the number of vertices and the number of vertices being $\{0,1\}$-valued.

Table 1: Number of vertices of the $\mathcal{FM}^k(X)$ polytopes.

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	4				
3	3	9	18			
4	4	16	303	166		
5	5	25	584740	407201	7579	
6	6	36	?	?	232871070690	7828352

Table 2: Number of $\{0,1\}$-valued vertices of the $\mathcal{FM}^k(X)$ polytopes.

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	4				
3	3	9	18			
4	4	16	68	166		
5	5	25	195	1855	7579	
6	6	36	456	10986	1322954	7828352

To compute these numbers for the cases with $n \leq 5$ we repeatedly use Theorem 5 to calculate the vertices of the intermediate polytopes. This process is very time-consuming and, in fact, took several weeks in a 2.1GHz PC to complete the calculations in the case $n = 5$. One of the main difficulties is that we lack criteria to quickly test adjacency (except for general fuzzy measures, where we can use Theorem 4). The other main problem is that the number of extreme points grows very fast, making the number of comparisons needed grow even faster. All this makes unfeasible a direct approach to calculate the number of vertices of the k-additive measures when $n = 6$.

However, it is possible to use the results on the isometries of the polytope of fuzzy measures (see Section 2) to at least obtain the number of vertices of the 5-additive measures when $n = 6$. Just observe that if σ is a permutation of X then S_σ, the symmetry induced by σ (Definition 11), is an isometry that fixes u_X and if μ is a fuzzy measure then $m_\mu(X)$ and $m_{S_\sigma(\mu)}(X)$ are both either positive, negative or zero, by Eq. (1). Also, as S_σ is an isometry, if μ_1 and μ_2 are adjacent vertices, then so are $S_\sigma(\mu_1)$ and $S_\sigma(\mu_2)$. Consequently, if $\lambda\mu_1 + (1-\lambda)\mu_2$ is a vertex of the $n-1$-additive measures then so is $\lambda S_\sigma(\mu_1) + (1-\lambda)S_\sigma(\mu_2)$.

Take μ, μ' two $\{0,1\}$-valued measures such that there exists a symmetry S_σ for which $S_\sigma(\mu) = \mu'$. If μ_1 is a $\{0,1\}$-valued measure adjacent to μ and $\alpha \in (0,1)$ such that

$$\alpha\mu + (1-\alpha)\mu_1 \in \mathcal{FM}^{n-1}(X),$$

then

$$S_\sigma(\alpha\mu + (1-\alpha)\mu_1) = \alpha S_\sigma(\mu) + (1-\alpha)S_\sigma(\mu_1)$$

also belongs to $\mathcal{FM}^{n-1}(X)$. Consequently, every vertex of $\mathcal{FM}^{n-1}(X)$ obtained from μ' is the image under S_σ of another vertex obtained from μ.

Thus, if μ and μ' are taken one into the other by some symmetry S_σ (i.e., they are in the same orbit under the action of the group of symmetries), then they will generate exactly the same number of vertices of $\mathcal{FM}^{n-1}(X)$. Therefore, in order to count the number of vertices of the $n-1$-additive measures, we can proceed as follows:

1. List all the vertices of the general fuzzy measures.
2. Classify these vertices according to the sign of their Möbius transform on X.
3. Calculate the orbits of the vertices with positive Möbius on X.
4. Pick a representative of each orbit and test its adjacency to the vertices with negative Möbius on X.
5. Multiply the number of adjacent vertices by the size of the orbit, and sum all the values obtained.
6. Add to this value the number of vertices whose Möbius transform on X is 0.

After applying these methods in the case where $n = 6$ we obtained 1322954 vertices with zero Möbius transform on X,

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3252699 with negative Möbius and also 3252699 with positive Möbius. These last vertices are organized into 6806 different orbits. After 20 hours of computation on a 2.1GHz PC, the number of non $\{0,1\}$-valued vertices of $\mathcal{FM}^5(X)$ with $|X| = 6$ was found to be 232869747736, for a total of 232871070690 vertices.

The figures in Tables 1 and 2 seem to suggest that the number of non $\{0,1\}$-valued vertices of $\mathcal{FM}^k(X)$ for $k = 3, ..., n-1$ grows much more quickly than the number of extreme points of $\mathcal{FM}(X)$. In the following Section we show that, in fact, this is the case when $k = n-1$.

4 On the number of vertices of the $n-1$-additive measures

In this section we study the asymptotic behavior of the ratio of the number of vertices of $\mathcal{FM}^{n-1}(X)$ to the number of vertices of $\mathcal{FM}(X)$. We will denote by D_n the number of vertices of $\mathcal{FM}(X)$ (with $n = |X|$), by A_n the number of vertices of $\mathcal{FM}^{n-1}(X)$, and by B_n the number of vertices of $\mathcal{FM}(X)$ which are not in $\mathcal{FM}^{n-1}(X)$ and whose minimal sets all have size at least $\lceil \frac{n-3}{2} \rceil$.

Lemma 2 *It holds that*

$$\lim_{n\to\infty} \frac{B_n}{D_n} > 0$$

Sketch of Proof:

The proof of the Theorem rests on considering, for even n, the set M_0^n of vertices μ of $\mathcal{FM}(X)$ such that:

1. every minimal set of μ has size $\frac{n}{2}-1$, $\frac{n}{2}$ or $\frac{n}{2}+1$,
2. $\mu(A) = 1$ for every A of size greater than $\frac{n}{2}+1$,
3. $\mu(A) = 0$ for every A of size less than $\frac{n}{2}-1$,
4. the number of minimal sets of μ of size $\frac{n}{2}-1$ is at most $2^{\frac{n}{2}}$.

In [32] it is proven that the ratio of the number of measures in M_0^n to the total number of measures does not tend to zero when n tends to infinity. It can also be seen that the ratio of non $(n-1)$-additive measures in M_0^n does not tend to 0 when n tends to infinity and the Lemma follows (for even n).

For odd n, the argument is similar, but considering the set M_1^n of vertices μ of $\mathcal{FM}(X)$ such that:

1. every minimal set of μ has size $\frac{n-3}{2}$, $\frac{n-1}{2}$ or $\frac{n+1}{2}$,
2. $\mu(A) = 1$ for every A of size greater than $\frac{n+1}{2}$,
3. $\mu(A) = 0$ for every A of size less than $\frac{n-3}{2}$,
4. the number of minimal sets of μ of size $\frac{n-3}{2}$ is at most $2^{\frac{n}{2}}$.

■

We can now state the main Theorem of the section.

Theorem 7 *There exist $k > 0$ and n_0 such that*

$$A_n > k\frac{2^n}{\sqrt{n}}D_n$$

for all $n \geq n_0$.

Sketch of Proof:

Denote $q = \lceil \frac{n-3}{2} \rceil$. Consider n such that $q > 4$. Consider $\mathcal{P}_1$ the collection of all partitions P of X such that

1. exactly one set in P has size $q-2$,
2. the rest of the sets in P are singletons,

and $\mathcal{P}_2$ the collection of all partitions P of X such that

1. exactly one set in P has size $q-3$,
2. the rest of the sets in P are singletons.

Consider μ a vertex of $\mathcal{FM}(X)$ not in $\mathcal{FM}^{n-1}(X)$ whose minimal sets all have size at least q. Consider P in $\mathcal{P}_1 \cup \mathcal{P}_2$ and μ' the measure whose collection of minimal sets is P. It is possible to show, with the help of Theorem 4, that μ and μ' are adjacent.

The size of $\mathcal{P}_1$ is $\binom{n}{q-2}$ and the size of $\mathcal{P}_2$ is $\binom{n}{q-3}$. Also, the sizes of the partitions in $\mathcal{P}_1$ and of the partitions in $\mathcal{P}_2$ have different parity, so their Möbius transform on X will be 1 in one case and -1 in the other (see Proposition 4 in [23]). Since μ is not $(n-1)$-additive and it is adjacent to every μ' whose minimal sets are in $\mathcal{P}_1 \cup \mathcal{P}_2$, then the convex combinations of μ with all such μ' generate at least $\binom{n}{q-3}$ vertices of $\mathcal{FM}^{n-1}(X)$. It follows that

$$A_n \geq \binom{n}{q-3} B_n$$

for all n such that $q > 4$.

From Lemma 2 we know that there exist $k_1 > 0$ and n_1 such that

$$\frac{B_n}{D_n} > k_1$$

for all $n \geq n_1$. Then, it is easy to prove that there exists $k > 0$ such that for n big enough

$$A_n \geq k\frac{2^n}{\sqrt{n}}D_n$$

■

This Theorem shows that the growth of the number of vertices of $\mathcal{FM}^{n-1}(X)$ is much faster than that of $\mathcal{FM}(X)$. Theorem 5 together with the values on Table 1 hint that this may be the case also for other polytopes $\mathcal{FM}^k(X)$ with k big enough. This suggests that, despite the simple interpretation, k-additive measures have a complex structure and the choice of the value of k should be made with care.

5 Conclusions and open problems

In this paper we have studied the vertices of the polytope of k-additive measures. We have provided a description of these vertices and a algorithm to compute them. We have also shown some examples of the results of the computation of this algorithm. Finally, we have given a lower bound on the number of vertices of the polytope of fuzzy measures which are at most $(n-1)$-additive, showing that this number grows much faster than the number of vertices of the general fuzzy measures. Similar results for other values of k seem plausible and deeper investigation of the asymptotical behavior on

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those cases might provide further insights on the polytopes $\mathcal{FM}^k(X)$.

The results presented here show that, in general, the structure of $\mathcal{FM}^k(X)$ is more complex than that of the general fuzzy measures and of the p-symmetric measures (cf.[29, 30]). For instance, knowledge of a wide number of intermediate polytopes was needed in order to compute the extreme points of the k-additive measures, making this computation very time consuming. A study of conditions for the adjacency in the polytope of k-additive measures (and in the intermediate ones) could help to decrease this computing time. Knowing the group of isometries of the polytopes may also be useful.

We also intend to investigate subfamilies of the k-additive measures which retain the modeling power of these measures. For instance, it is interesting to restrict the vertices to those which are $\{0, 1\}$-valued and study the resulting polytope.

Finally, in order to obtain a suitable initial population (for measure identification with a genetic algorithm) it would be useful to study methods for the random generation of k-additive measures.

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