On vertices of the k-additive monotone core

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Abstract— Given a capacity, the set of dominating k-additive capacities is a convex polytope; thus, it is defined by its vertices. In this paper we deal with the problem of deriving a procedure to obtain such vertices in the line of the results of Shapley and Ichiishi for the additive case. We propose an algorithm to determine the vertices of the k-additive monotone core. Then, we characterize the vertices of the n-additive core and finally, we explore the possible translations for the k-additive case.

Keywords— Capacities, k-additivity, dominance, core.

1 Introduction

One of the main problems of cooperative game theory is to define a solution of a game μ, that is, supposing that all players join the grand coalition X, an *imputation* to each player represents a sharing of the total worth of the game $\mu(X)$. In the case of finite games of n players, an imputation can be written as a n-tuple $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = \mu(X)$. Of course, some rationality criterion should prevail when defining the sharing.

In this respect, the core is perhaps the most popular solution of a game. It is a well known fact that the core is nonempty if and only if the game is balanced [1]. However, there are games whose core is empty. It is then necessary to give an alternative solution. In this sense, many possibilities have been proposed in the literature, as the dominance core stable sets, Shapley index, the nucleolus, etc. (see e.g. [2]).

On the other hand, Grabisch has defined in [3] the concept of k-additive capacities, for a fixed value $1 \leq k \leq n$. These capacities generalize the concept of probability and they fill the gap between probabilities and general capacities. Moreover, as they are defined in terms of the Möbius transform and this transform can be applied to the characteristic function of any game (not necessarily monotone), the concept of k-additivity can be extended to games as well.

In a previous paper we have defined the so-called k-additive core. The basic idea is to remark that an imputation is nothing other than an additive game, and if the core is empty, we may allow to search for games more general than additive ones, namely k-additive games, dominating the game. We have presented a generalization of balanced games, the k-balanced games, that are those admitting a dominating k-additive game and no dominating $(k-1)$-additive game.

We have seen that for general games, any game is either balanced or 2-balanced. Moreover, the 2-additive core is not a bounded polytope but an unbounded convex polyhedron. However, when dealing with capacities, it makes sense to study the k-additive monotone core and it can be easily seen that in this particular case it is a convex polytope, whence it can be defined through describing its vertices. This paper studies these vertices. In the framework of Game Theory, it has been solved for the (1-additive) core by Shapley and Ichiishi. The vertices of the $(n-1)$-additive core has been obtained in [4].

Moreover, there are other fields in which it is interesting to find the set of probabilities dominating a capacity. For instance, Dempster [5] and Shafer [6] have proposed a representation of uncertainty based on a "lower probability" or "degree of belief", respectively, to every event. Their model needs a lower probability function, usually non-additive but having a weaker property: it is a belief function [6]. This requirement is perfectly justified in some situations (see [5]). The general form of lower probabilities has been studied by several authors (see e.g. [7, 8]). Moreover, in many decision problems, in which we have not enough information, decision makers often feel that they are only able to assign an interval value for the probability of events. In other words, they do not know the real probability distribution but there exists a set of probabilities compatible with the available information. Let us call this set of all compatible probabilities $\mathcal{P}_1$ and let us define $\mu = \inf_{P \in \mathcal{P}_1} P$; then, μ is a capacity (but not necessarily a belief function [9]); μ is called "coherent lower probability", and it is the natural "lower probability function". Of course, if P' is a probability measure dominating μ, it is clear that $E_{P'}(f) \geq \mathcal{C}_\mu(f)$, for any function f, where $\mathcal{C}_\mu$ represents Choquet integral [10]. Chateauneuf and Jaffray use this fact and that $\mu \leq P$, $\forall P \in \mathcal{P}_1$ in [11] to obtain an easy method for computing a lower bound of $\inf_{P \in \mathcal{P}_1} E_P(f)$ whenever μ is 2-monotone. Their method is based on obtaining the set of all probability distributions dominating μ. The same can be done for obtaining an upper bound. In this case, we can find a similar motivation for studying the set of all k-additive capacities dominating a capacity.

The paper is organized as follows: In next section, we give the basic concepts about k-additive capacities and about the set of dominating probabilities. Next, in Section 3 we provide an algorithm for obtaining the set of all k-additive dominating capacities. Section 4 is devoted to characterize the vertices for the n-additive case and, in Section 5, we deal with possible generalizations for the k-additive case.

2 Basic concepts

We will use the following notations: we suppose a finite universal set with n elements, $X = \{1, ..., n\}$. Subsets of X are

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denoted by capital letters A, B, and so on. The set of subsets of X is denoted by $\mathcal{P}(X)$, and the set of subsets whose cardinality is maximum k is denoted by $\mathcal{P}^k(X)$.

Definition 1 *[12] A* **game** *over X is a mapping $\mu : \mathcal{P}(X) \to \mathbb{R}$ (called* **characteristic function***) satisfying $\mu(\emptyset) = 0$.*

If, in addition,

1. *μ satisfies $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$, the game μ is said to be* **monotone***;*
2. *μ satisfies $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \subseteq X$, $A \cap B = \emptyset$, the game is said to be* **additive***;*
3. *μ satisfies $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$, for all $A, B \subseteq X$, the game is said to be* **convex**.
4. *μ satisfies*
$$\mu(\bigcup_{i=1}^{k} A_i) \geq \sum_{\substack{K \subseteq \{1,\ldots,k\} \\ K \neq \emptyset}} (-1)^{|K|+1} \mu(\bigcap_{j \in K} A_j) \quad (1)$$
for any family of k subsets $A_1, \ldots A_k$, $k \geq 2$, the game is said to be **k-monotone**.

Definition 2 *A* **non-additive measure** *[13] or* **capacity** *[10] or* **fuzzy measure** *[14] μ over X is a monotone game with $\mu(X) = 1$.*

Note that any monotone game can be equivalently defined through a capacity. The set of all capacities on X is a convex polytope, that we will denote $\mathcal{FM}(X)$.

Definition 3 *[15] Let μ be a game on X. The* **Möbius transform (or inverse)** *of μ is a set function on X defined by*

$$m_\mu(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \, \forall A \subseteq X. \quad (2)$$

The Möbius transform given, the original characteristic function can be recovered through the *Zeta transform* [11]:

$$\mu(A) = \sum_{B \subseteq A} m(B). \quad (3)$$

Let us turn to the concept of k-additivity. In order to define a capacity, $2^n - 2$ values are necessary. The number of coefficients grows exponentially with n, and so does the complexity of the problem. This drawback reduces considerably the practical use of capacities. Then, some subfamilies of capacities have been defined in an attempt to reduce complexity. In this paper we will use k-additive capacities.

Definition 4 *[16] A game μ is said to be* **k-order additive** *or* **k-additive** *for some $k \in \{1, \ldots, n\}$ if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$, and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.*

In this sense, a probability is just a 1-additive capacity [16]. Thus, k-additive capacities generalize probabilities, that are very restrictive in many situations as they do not allow interactions between the elements of X. They fill the gap between probabilities and general non-additive capacities. We will denote by $\mathcal{FM}^k(X)$ (resp. $\mathcal{G}^k(X)$) the set of all k'-additive capacities (resp. games) with $k' \leq k$.

Let us introduce the concept of k-additive monotone core.

Definition 5 *Let $\mu\mu^*$ be two games. We say μ^** **dominates** *μ, and we denote it $\mu^* \geq \mu$, if and only if*

$$\mu^*(A) \geq \mu(A), \, \forall A \subset X, \mu^*(X) = \mu(X). \quad (4)$$

One of the main problems of cooperative game theory is to define a solution of a game ν, that is, supposing that all players join the grand coalition X, an *imputation* to each player represents a sharing of the total worth of the game $\nu(X)$. In the case of finite games of n players, an imputation can be written as a n-tuple $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^n x_i = \nu(X)$. Of course, some rationality criterion should prevail when defining the sharing.

Definition 6 *Let μ be a game. We say that a vector : $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is an* **imputation** *for μ if it satisfies*

$$\sum_{i=1}^{n} x_i = \mu(X). \quad (5)$$

An imputation represents a possible pay-off for the players, i.e. supposing that all players agree to form the grand coalition, it provides a possible sharing of the value $\mu(X)$ among the players.

Remark 1 *For any $\mathbf{x} \in \mathbb{R}^n$, it is convenient to use the notation $x(A) := \sum_{i \in A} x_i$, for all $A \subseteq X$, with the convention $x(\emptyset) = 0$. Thus, x identifies with an additive game for which the values on singletons are x_i.*

The value x_i is the asset player i receives when sharing $\mu(X)$. Suppose that the imputation satisfies $x(A) \geq \mu(A)$, for all $A \subseteq X$. If this is the case, all players should agree with their pay-off, as if they try to form other coalition different of X, the corresponding value for the coalition would be worse than the value the coalition obtains with the additive game x. In other words, any such $(x_1, \ldots, x_n)$ is a possible satisfactory imputation for all players.

Definition 7 *[17] Let μ be a game. The* **core** *of μ, denoted by $\mathcal{C}(\mu)$, is defined by*

$$\mathcal{C}(\mu) := \{x \in \mathbb{R}^n \mid x(A) \geq \mu(A), \, \forall A \subseteq X, x(X) = \mu(X)\}.$$

Since by Remark 1 any $x \in \mathbb{R}^n$ induces an additive game, the core can be equivalently defined as the set of additive games dominating μ. When the core is nonempty, it is usually taken as the solution of the game. Note that for the case of the core, given a dominating additive game, the value x_i coincides with $m(i)$. However, there are games with an empty core. Then, the following definition arises:

Definition 8 *[12] A game μ is* **balanced** *if $\mathcal{C}(\mu) \neq \emptyset$.*

For the special case of μ being a capacity, if $(x_1, \ldots, x_n)$ is in the core, it follows that $(x_1, \ldots, x_n)$ determines a probability distribution on X dominating μ. Thus, in this case, $\mathcal{C}(\mu)$ coincides with the set of all probabilities dominating μ.

When non-empty, the core is a convex polytope and its vertices are known when the game is convex.

Definition 9 *A* **maximal chain** *in 2^X is a sequence of subsets $A_0 := \emptyset, A_1, \ldots, A_{n-1}, A_n := X$ such that $A_i \subset A_{i+1}$, $i = 0, \ldots, n-1$. The set of maximal chains of 2^X is denoted by $\mathcal{M}(2^X)$.*

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To each maximal chain $C := \{\emptyset, A_1, \ldots, A_n = X\}$ in $\mathcal{M}(2^X)$ corresponds a unique permutation σ on X such that $A_1 = \sigma(1)$, $A_2 \setminus A_1 = \sigma(2)$, ..., $A_n \setminus A_{n-1} = \sigma(n)$. The set of all permutations over X is denoted by $\mathfrak{S}(X)$. Let μ be a capacity. Each permutation σ (or maximal chain C) induces an additive capacity ϕ^σ (or ϕ^C) on X defined by:

$$\phi^\sigma(\{\sigma(i)\}) := \mu(\{\sigma(1), \ldots, \sigma(i)\}) - \mu(\{\sigma(1), \ldots, \sigma(i-1)\}) \tag{6}$$

or

$$\phi^C(\{\sigma(i)\}) := \mu(A_i) - \mu(A_{i-1}), \quad \forall i \in X, \tag{7}$$

with the above notation.

Theorem 1 *The following propositions are equivalent.*

1. *μ is a convex capacity.*
2. *All additive capacities ϕ^σ, $\sigma \in \mathfrak{S}(X)$, belong to the core of μ.*
3. $\mathcal{C}(\mu) = \mathrm{co}(\{\phi^\sigma\}_{\sigma\in\mathfrak{S}(X)})$.
4. $\mathrm{ext}(\mathcal{C}(\mu)) = \{\phi^\sigma\}_{\sigma\in\mathfrak{S}(X)}$,

where $\mathrm{co}(\cdot)$ and $\mathrm{ext}(\cdot)$ denote respectively the convex hull of some set, and the extreme points of some convex set.

(i) ⇒ (ii) and (i) ⇒ (iv) are due to Shapley [17], while (ii) ⇒ (i) was proved by Ichiishi [18].

In a previous work [19] we have defined the so-call k-additive monotone core.

Definition 10 *For some integer $1 \le k \le n$, the k-**additive monotone core** of a capacity μ is defined by:*

$$\mathcal{MC}^k(\mu) := \{\phi \in \mathcal{FM}^k(X) \mid \phi(A) \ge \mu(A), \quad \forall A \subseteq X\}.$$

If non-empty, it is easy to see that $\mathcal{MC}^k(\mu)$ is a convex polytope. In next sections we will study its vertices, i.e. the capacities such that they cannot be put as a convex combination of two other capacities in the polytope.

3 An algorithm for determining vertices of the k-additive monotone core

Take $\mu \in \mathcal{FM}(X)$. The polytope $\mathcal{MC}^k(\mu)$ can be seen as a subpolytope of $\mathcal{FM}^k(X)$, given by the additional constraints

$$\mu^*(A) \ge \mu(A), \forall A \subseteq X. \tag{8}$$

I.e. we restrict the polytope to the measures $\mu^* \in \mathcal{FM}^k(X)$ dominating μ. Thus, we propose the following algorithm to determine its vertices:

- Initialization: $\mathcal{FM}^k(X)$.
- Take $A \subseteq X$ and add the constraint $\mu^*(A) \ge \mu(A)$.
- Obtain the vertices and the adjacency structure (i.e. whether two vertices are in an edge) of the new polytope.
- Repeat for any $A \subseteq X, A \ne X, \emptyset$.

Let us analyze a step. We will denote by $\mathcal{F}_1$ the polytope before introducing the new constraint $\mu^*(A) \ge \mu(A)$ and by $\mathcal{F}_2$ the resulting polytope. For $\mathcal{F}_1$, we assume that we know its vertices and its adjacency structure. The vertices of $\mathcal{F}_2$ are:

- Vertices of $\mathcal{F}_1$ satisfying the new constraint.
- New vertices, coming from the intersection of the hyperplane defined by $\mu^*(A) = \mu(A)$ and $\mathcal{F}_1$. It can be easily proved that these new vertices are in edges of $\mathcal{F}_1$.

Thus, in order to determine $\mathcal{F}_2$, it suffices to know the vertices and the edges (whether two vertices are adjacent) of $\mathcal{F}_1$.

In order to apply this procedure again, let us determine the adjacency structure of $\mathcal{F}_2$. Consider μ_1, μ_2 two vertices of $\mathcal{F}_2$. We have several cases:

- μ_1, μ_2 are vertices of $\mathcal{F}_1$.
- μ_1 is a vertex of $\mathcal{F}_1$ but μ_2 is not.
- Neither μ_1 nor μ_2 are vertices of $\mathcal{F}_1$.

We will study whether μ_1 and μ_2 are adjacent vertices in each situation.

Consider μ_1, μ_2 two vertices of $\mathcal{F}_2$ that are also vertices of $\mathcal{F}_1$. If they are adjacent vertices in $\mathcal{F}_1$, then they are adjacent in $\mathcal{F}_2$, as $\mathcal{F}_2$ is a subpolytope of $\mathcal{F}_1$ and we are done.

Lemma 1 *Assume μ_1, μ_2 are not adjacent vertices in $\mathcal{F}_1$. Then, if μ_1, μ_2 are adjacent vertices in $\mathcal{F}_2$, they satisfy $\mu_1(A) = \mu(A) = \mu_2(A)$.*

Moreover, the following holds:

Lemma 2 *Consider μ_1, μ_2 two vertices of $\mathcal{F}_1$ and assume they are adjacent vertices in $\mathcal{F}_2$. Then, if μ_1, μ_2 are not adjacent in $\mathcal{F}_1$, necessarily $\mu_1(A) = \mu(A) = \mu_2(A)$.*

Remark 2 *It can be proven that if μ_1, μ_2 are in the conditions of the previous lemma, then they are in a facet of dimension 2 of $\mathcal{F}_1$.*

Moreover, the intersection of this facet with the hyperplane $\mu^*(A) = \mu(A)$ is exactly the segment $[\mu_1, \mu_2]$. Otherwise, if we can find μ_3 outside the segment and in the facet satisfying $\mu_3(A) = \mu(A)$, we can build two linearly independent vectors in the facet and in the hyperplane, whence the facet is contained in the hyperplane and thus, the facet is contained in $\mathcal{F}_2$. But this would imply that, as μ_1, μ_2 are not adjacent in $\mathcal{F}_1$, they are not adjacent in $\mathcal{F}_2$, a contradiction.

Let us now turn to the case in which μ_1 is a vertex of $\mathcal{F}_1$ but μ_2 is not. The conditions for μ_1, μ_2 being adjacent vertices in $\mathcal{F}_2$ are given in next lemma.

Lemma 3 *Consider μ_1, μ_2 two vertices of $\mathcal{F}_2$ and suppose μ_1 is a vertex of $\mathcal{F}_1$ and μ_2 is not. If they are adjacent in $\mathcal{F}_2$, then:*

- *If $\mu_1(A) > \mu(A)$, then μ_2 is in an edge starting in μ_1.*
- *If $\mu_1(A) = \mu(A)$, then μ_1 and μ_2 are in a facet of $\mathcal{F}_1$ of dimension 2. Indeed, the intersection of the facet with the hyperplane $\mu^*(A) = \mu(A)$ is the segment $[\mu_1, \mu_2]$.*

Moreover, the following can be proved:

Lemma 4 *Consider μ_2 a vertex of $\mathcal{F}_2$ such that μ_2 is not a vertex of $\mathcal{F}_1$. Then, there exists exactly one vertex μ_1 of $\mathcal{F}_1$ such that $\mu_1(A) > \mu(A)$ that is adjacent to μ_2 in $\mathcal{F}_2$.*

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Finally, let us consider the case of μ_1, μ_2 being two vertices of $\mathcal{F}_2$ that are not vertices of $\mathcal{F}_1$.

Lemma 5 *Suppose μ_1, μ_2 are vertices of $\mathcal{F}_2$ that are not vertices of $\mathcal{F}_1$. If they are adjacent vertices of $\mathcal{F}_2$, they are in the same facet of dimension 2 of $\mathcal{F}_1$. Indeed, the intersection of the facet with the hyperplane $\mu^*(A) = \mu(A)$ is the segment $[\mu_1, \mu_2]$.*

As a final remark about this section note that, in order to apply this procedure, it is necessary to know the adjacency structure of the polytope $\mathcal{FM}^k(X)$. As proved in [20, 21], the problem of determining non-adjacency of vertices of a polytope is, in some cases, NP-complete. The vertices of $\mathcal{FM}(X)$ are $\{0,1\}$-valued measures [22]. In [23] a characterization of the adjacency in $\mathcal{FM}(X) = \mathcal{FM}^n(X)$ that allows us to check whether two vertices are adjacent in quadratic time has been obtained. The adjacency structure of $\mathcal{FM}^1(X)$ and $\mathcal{FM}^2(X)$ is also known. However, the structure of $\mathcal{FM}^k(X)$ for other values of k is more complicated [24] and the adjacency structure is not known (indeed, the vertices of the polytope have not been obtained yet). In this last case, we are forced to apply a similar algorithm with the additional constraints $m(A) = 0$ before applying the procedure.

4 The set $\mathcal{MC}^n(\mu)$.

The procedure stated in the previous section can be very time-consuming and thus unfeasible in practice for big values of $|X|$. Thus, it is interesting to look for a characterization of the vertices of the k-additive core; in this line we have the results of Shapley and Ichiishi [17, 18] for $\mathcal{C}^1(\mu)$ and the results in [4] for $\mathcal{C}^{n-1}(\mu)$ (the $(n-1)$-additive core, not restricted to capacities). In this section we provide a characterization of vertices of $\mathcal{MC}^n(\mu)$. We consider the following procedure.

- Let $\prec$ be an order on $\mathcal{P}(X) \setminus \{X, \emptyset\}$. This order allows us to rank the different subsets of X,

$$A_1 \prec A_2 \prec ... \prec A_{2^n-2}. \quad (9)$$

- Next, take a partition $\mathcal{P} = \{\mathcal{U}, \mathcal{L}\}$ on $\mathcal{P}(X) \setminus \{X, \emptyset\}$, where $\mathcal{U}$ or $\mathcal{L}$ could be empty.

- **Initializing step**: For $\prec$ and $\mathcal{P}$ fixed, let us define

$$\overline{\mu}^0(A_i) = 1, \underline{\mu}^0(A_i) = \mu(A_i), \forall A_i. \quad (10)$$

- **Iterating step**: For $i = 1$ until $i = 2^n - 2$, do:
  - If $A_i \in \mathcal{U}$, then assign

$$\mu_{\prec,\mathcal{P}}(A_i) = \overline{\mu}^{i-1}(A_i). \quad (11)$$

Redefine:
For $\underline{\mu}^i$, we put

$$\underline{\mu}^i(B) = \max\{\overline{\mu}^{i-1}(A_i), \underline{\mu}^{i-1}(B)\}, \text{ if } A_i \subseteq B \quad (12)$$

$$\underline{\mu}^i(B) = \underline{\mu}^{i-1}(B), \text{ otherwise.} \quad (13)$$

For $\overline{\mu}^i$, we put

$$\overline{\mu}^i(B) = \overline{\mu}^{i-1}(B), \forall B \subset X. \quad (14)$$

  - If $A_i \in \mathcal{L}$, then assign

$$\mu_{\prec,\mathcal{P}}(A_i) = \underline{\mu}^{i-1}(A_i). \quad (15)$$

Redefine:
For $\overline{\mu}^i$, we put

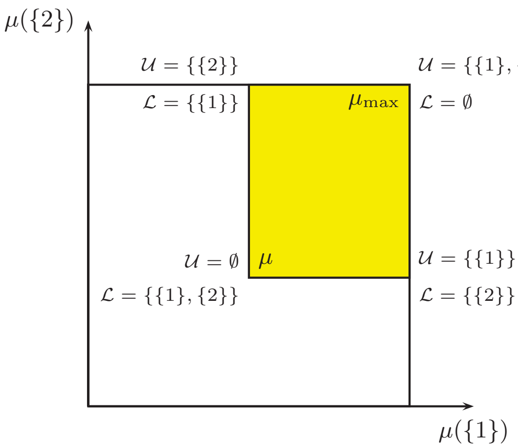
$$\overline{\mu}^i(B) = \min\{\underline{\mu}^{i-1}(A_i), \overline{\mu}^{i-1}(B)\}, \text{ if } B \subseteq A_i \quad (16)$$

$$\overline{\mu}^i(B) = \overline{\mu}^{i-1}(B), \text{ otherwise.} \quad (17)$$

For $\underline{\mu}^i$, we put

$$\underline{\mu}^i(B) = \underline{\mu}^{i-1}(B), \forall B \subset X. \quad (18)$$

The idea of the procedure is the following: If we are in step i, the values for $A_1, ..., A_{i-1}$ are fixed. For A_i, if $A_i \in \mathcal{U}$, we assign to $\mu_{\prec,\mathcal{P}}(A_i)$ the biggest possible value keeping dominance, which is $\overline{\mu}^i(A_i)$. Similarly, if $A_i \in \mathcal{L}$, we assign to $\mu_{\prec,\mathcal{P}}(A_i)$ the smallest possible value keeping dominance, which is $\underline{\mu}^i(A_i)$. Once the value of $\mu_{\prec,\mathcal{P}}(A_i)$ is fixed, we need to actualize the lower and upper bounds for $A_j, j > i$. These lower and upper bounds are stored in $\underline{\mu}^{i+1}$ and $\overline{\mu}^{i+1}$, respectively. The Figure below explains the performance of the algorithm for the special case of $|X| = 2$. In it, we can see that $\mathcal{MC}^2(\mu)$ has four vertices. Note that μ itself is always a vertex of $\mathcal{MC}^2(\mu)$, corresponding to the case $\mathcal{L} = \mathcal{P}(X) \setminus \{X, \emptyset\}$ and $\mathcal{U} = \emptyset$. Thus, for the n-additive case, $\mu \in \mathcal{MC}^n(\mu)$ and it is the bottom element of this set. Similarly, the measure attaining value 1 for every subset is another vertex of $\mathcal{MC}^2(\mu)$, corresponding to the case $\mathcal{U} = \mathcal{P}(X) \setminus \{X, \emptyset\}$ and $\mathcal{L} = \emptyset$. Thus, for the n-additive case, it is the top element of the set. This also holds when $|X| > 2$.



Note also that this procedure generalizes for the n-additive case the Shapley-Ichiishi theorem for probabilities. In our case, the total order on $\mathcal{P}(X) \setminus \{\emptyset, X\}$ plays the role of chains; the value that is assigned to a singleton in the Shapley-Ichiishi theorem is indeed the upper and lower bound for this value in order to keep dominance. However, in our case we need two "chains" instead of one because lower and upper bounds are not the same for the general case.

Finally, remark that for the general case we do not need to impose any additional condition on μ, while for the set $\mathcal{C}(\mu) = \mathcal{MC}^1(\mu)$ convexity is required.

In next results we will prove that the function $\mu_{\prec,\mathcal{P}}$ obtained through this procedure is a vertex of $\mathcal{MC}^n(\mu)$ and that any vertex can be obtained through a suitable choice of $\prec$ and $\mathcal{P}$.

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Proposition 1 $\mu_{\prec,\mathcal{P}} \in \mathcal{MC}^n(\mu)$.

The proof of this result is based on the following lemmas, that provide us with additional properties of $\{\overline{\mu}^i(B)\}_{i=0}^{2^n-2}$ and $\{\underline{\mu}^i(B)\}_{i=0}^{2^n-2}$.

Lemma 6 *For any $B \subset X$, the sequence $\{\underline{\mu}^i(B)\}_{i=1}^{2^n-2}$ is nondecreasing. Similarly, the sequence $\{\overline{\mu}^i(B)\}_{i=0}^{2^n-2}$ is non-increasing.*

Lemma 7 $\underline{\mu}^i, \overline{\mu}^i \in \mathcal{FM}(X), \forall i = 0, ..., 2^n - 2$.

Lemma 8 $\overline{\mu}^i \geq \underline{\mu}^i, \forall i = 0, ..., 2^n - 2$.

Moreover, the following holds.

Proposition 2 *$\mu_{\prec,\mathcal{P}}$ is a vertex of $\mathcal{MC}^n(\mu)$.*

Finally, it can be seen that all the vertices can be derived from this procedure.

Proposition 3 *If μ^* is a vertex of $\mathcal{MC}^n(\mu)$, there exists an order $\prec$ and a partition $\mathcal{P}$ of $\mathcal{P}(X) \setminus \{X, \emptyset\}$ such that $\mu^* = \mu_{\prec,\mathcal{P}}$.*

Note that these results allows us to derive an upper bound for the number of vertices of $\mathcal{MC}^n(\mu)$.

Proposition 4 *The number of vertices of $\mathcal{MC}^n(\mu)$ is bounded by 2^{2^n-2}.*

5 The case of $\mathcal{MC}^k(\mu)$

In this section we treat the general k-additive case. The basic idea is to translate the results of the previous section to this case. In order to translate the results, we will need to solve two new problems:

- For a fixed value of $\mu(A)$, the possible lower and upper bounds of $\mu(B), B \neq A$ are not trivial, as it happened for the n-additive case. Moreover, if we are dealing with $\mathcal{MC}^k(\mu)$, it could be the case that $\mu \notin \mathcal{FM}^k(X)$. Thus, in the k-additive case, it could be the case that no such bottom element for $\mathcal{MC}^k(\mu)$ exists, so we cannot define $\underline{\mu}^0$. Similarly, the measure attaining value 1 for every subset is no longer in $\mathcal{MC}^k(\mu)$, whence we cannot define $\overline{\mu}^0$.
- The structure of the polytope $\mathcal{FM}^k(X)$ is not known for $k \geq 3$. Indeed, it has been proved in [24] that there are vertices of $\mathcal{FM}^k(X), k \geq 3$ that are not $\{0,1\}$-valued measures; moreover, we do not know the vertices of the polytope. However, it could be the case that some of these vertices are in $\mathcal{MC}^k(\mu)$. How can they be characterized?

We will study in this section the particular case in which $\mu \in \mathcal{FM}^k(X)$. Notice that in the n-additive general case, this condition trivially holds. Of course, if $\mu \in \mathcal{FM}^k(X)$, then $\mathcal{MC}^k(\mu) \neq \emptyset$ and has a bottom element (μ itself). Notice again that this was the situation for the n-additive case.

- Let us consider a total order on $\mathcal{P}^k_*(X) := \mathcal{P}^k(X) \setminus \{\emptyset\}$. Notice that in the general case, we have considered an order on $\mathcal{P}(X) \setminus \{\emptyset, X\}$. This total order allows us to range the subsets in $\mathcal{P}^k_*(X)$: $A_1 \prec A_2 \prec ... \prec A_r$, where $r = \sum_{i=1}^k \binom{n}{i}$.
- Next, take a partition $\mathcal{P} = \{\mathcal{U}, \mathcal{L}\}$ on $\mathcal{P}(X) \setminus \{X, \emptyset\}$, where $\mathcal{U}$ or $\mathcal{L}$ could be empty.
- **Initializing step**: Let us define $\mu_0 := \mu$
- **Iterating step**: For $i = 1$ until $i = r$ do:
  - If $A_i \in \mathcal{L}$, then $\mu_i = \mu_{i-1}$.
  - Otherwise $A_i \in \mathcal{U}$. It is easy to see that $\mu_{i-1} \in \mathcal{FM}^k(X)$.

    Let us consider the subset given by
    $$\{\mu^* \in \mathcal{FM}^k(X) \,|\, \mu^* \geq \mu_{i-1}, \\ \mu^*(A_j) = \mu_{i-1}(A_j), j = 1, ..., i-1\}. \quad (19)$$
    As $\mathcal{FM}^k(X)$ is a polytope, so is this set. Thus, for A_i, it follows that $\mu^*(A_i)$ can vary in an interval $[\mu_{i-1}(A_i), s]$. Take the subset of measures in the set satisfying $\mu^*(A_i) = s$. Let us denote this subset by $\mathcal{A}_0$.

    As before, $\mathcal{A}_0$ is a polytope. Then, for $\mu^* \in \mathcal{A}_0$, the value $\mu^*(A_{i+1})$ can vary in an interval $[x_1^-, x_1^+]$. We define
    $$\mathcal{A}_1 := \{\mu^* \in \mathcal{A}_0 \,|\, \mu^*(A_{i+1}) = x_1^-\}. \quad (20)$$
    Now, the same can be done for $\mathcal{A}_1$ and considering A_{i+2}. We reiterate until $\mathcal{A}_j$ is just a singleton (notice that necessarily $\mathcal{A}_{r-i}$ is a singleton). This capacity is μ_i.

It can be checked that this algorithm coincides with the algorithm of the previous section for the n-additive case. Let us denote by μ_r the measure obtained in the last iteration.

Note that if $k = 1$ and μ is a probability, then the set $\mathcal{MC}^1(\mu) = \mathcal{C}(\mu) = \{\mu\}$ and the problem is trivial.

Now, the following can be proved:

Lemma 9 *If $\mu \in \mathcal{FM}^k(X)$, the measure μ_r obtained in the procedure is an extreme point of $\mathcal{MC}^k(\mu)$.*

However, this method does not obtain all the vertices of $\mathcal{MC}^k(\mu)$, as next example shows:

Example 1 *Consider $|X| = 4$ and the measure $u_{\{1,4\}}$ given by $u_{\{1,4\}}(A) = 1$ if $\{1,4\} \subseteq A$ and $u_{\{1,4\}}(A) = 0$ otherwise. Then, $u_{\{1,4\}} \in \mathcal{FM}^3(X)$. Consider the measure μ^* given by*

Subset	*1*	*2*	*3*	*4*	*1,2*	*1,3*	*1,4*
μ^*	*0*	*0*	*0*	*0*	*0.5*	*0.5*	*1*

Subset	*2,3*	*2,4*	*3,4*	*1,2,3*	*1,2,4*	*1,3,4*	*2,3,4*
μ^*	*0.5*	*0*	*0*	*0.5*	*1*	*1*	*1*

It has been proved in [24] that μ^* is an extreme point of $\mathcal{FM}^3(X)$. On the other hand, $\mu^* \geq u_{\{1,4\}}$, whence μ^* is an extreme point of $\mathcal{MC}^3(u_{\{1,4\}})$. Let us check that μ^* cannot be obtained through the previous algorithm, no matter the order considered.

Assume $A_1 \in \mathcal{U}$.

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- *If A_1 is a singleton, straightforward calculus shows that we can obtain $\mu_1(A_1) = 1$, whence μ^* could not be derived. The same holds if A_1 is a pair different from $\{1,4\}$ and $\{2,3\}$.*

- *Suppose $A_1 = \{2,3\}$ and consider the measure whose Möbius transform is given by*

$$m'(\{2,3\}) = 1, m'(\{1,4\}) = 1, m'(\{2,4\}) = 1. \quad (21)$$

In this case, we obtain a 3-additive measure and thus, it is possible to obtain $\mu_1(\{2,3\}) = 1$. Therefore, μ^* cannot be derived in this case.

- *If $A_1 = \{1,2,3\}$, then $m'(\{1,2\}) = 1$ gives $\mu_1(\{1,2,3\}) = 1$, whence μ^* cannot be derived in this case.*

- *For the other possibilities, we have $u_{\{1,4\}}(A_1) = 1$, whence the value is fixed.*

Consequently, it is not possible to obtain μ^* if $A_1 \in \mathcal{U}$. Thus, assume $A_1 \in \mathcal{L}$. This fixes $\mu_1(A_1) = u_{\{1,4\}}(A_1)$.

- *If $A_1 \in \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, then $\mu_1(A_1) = 0$, whence μ^* cannot be recovered.*

- *For other possibilities, we fix a value either 0 or 1.*

But then, we can repeat the process for A_2 with the same results. Thus, μ^* cannot be obtained through the procedure.

6 Conclusions

In this paper we have proposed an algorithm to obtain the vertices of the k-additive monotone core. This procedure can be applied to any value of k. However, it seems that it could be very time-consuming. Next, we have derived the vertices of the n-additive core; these results generalize the Shapley-Ichiishi theorem for the general case. Finally, we have treated the possible extensions for the k-additive case.

An important problem arising in the k-additive case is the number of vertices of the k-additive core. For the general n-additive case, the set of vertices of $\mathcal{FM}(X)$ coincides with the n-th Dedekind number; simulations carried on for the k-additive case seem to show that the number of vertices of $\mathcal{FM}^k(X)$ is even greater, due to vertices that are not $\{0,1\}$-valued. This problem could make unfeasible to store all the vertices of the k-additive core in some cases.

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