Resolution of fuzzy relation equations with sup-inf composition over complete Brouwerian lattices—a review *

Xue-ping Wang[1] Feng Sun[1]

1.Department of Mathematics, Sichuan Normal University
Chengdu, Sichuan 610066, People's Republic of China
Email: xpwang1@hotmail.com, sunfeng1005@163.com

Abstract— *This paper restates the results on fuzzy relation equations with sup-inf composition from the viewpoint of decomposition, presents a way to describe the solution set of fuzzy relation equations, and shows a necessary and sufficient condition, which partly answers the open problem for existence of minimal solutions over complete Brouwerian lattices.*

Keywords— Complete Brouwerian lattice, fuzzy relation equation, minimal solution, solution set, sup-inf composition.

1 Introduction and preliminaries

Let I, J be index sets and $A = (a_{ij})_{I\times J}$ be a coefficient matrix, $B = (b_i)^T_{i\in I}$ be a constant column vector (the sign "T" denotes the "transpose"). Then $A \odot X = B$ or

$$\bigvee_{j\in J} (a_{ij} \wedge x_j) = b_i, i \in I \tag{1}$$

is called a *fuzzy relation equation* assigned on a lattice L, where $\odot$ denotes the sup-inf composite operation, and all x_j, b_i, a_{ij}'s are in L. An X which satisfies (1) is called a *solution* of (1), the *solution set* of (1) is denoted by $\mathscr{X}_1 = \{X : A \odot X = B\}$. A special case of (1) is as follows: $A \odot X = b$ or

$$\bigvee_{j\in J} (a_j \wedge x_j) = b, \tag{2}$$

where $b \in L$, $A = (a_j)_{j\in J}$ is a row vector. Denote $\mathscr{X}_2 = \{X : A \odot X = b\}$ the solution set of (2).

The solvability of fuzzy relation equations in complete Brouwerian lattices was first proposed in connection with medical diagnosis problems in [24]. Sanchez [24] showed that every solvable fuzzy relation equation assigned on complete Brouwerian lattices has the greatest solution. Since then, the resolution of fuzzy relation equations over lattices has been a theme of continuous interest in fuzzy inference and fuzzy systems theory. A number of works in this area were published (see, e.g. [2,6,8,9,12,17,19,20,23,25,27,28,29,30,32,33,38]). Many results are obtained when L is a linear lattice (see, e.g. [3, 7, 13, 14, 15, 18, 26]). There are also many results considered fuzzy relation equations over semi-linear spaces (see [10, 16, 21, 22] for details), and many works dealt with the more general fuzzy relation equations on complete lattices, such as $sup_{j\in J}\mathscr{T}(a_j, x_j) = b$, where $\mathscr{T}$ is a t-norm (see [5]) or a pseudo-t-norm (see [11,34]) or a conjunctor (see [31,37]).

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This paper mainly focuses on the fuzzy relation equations over complete Brouwerian lattices.

We first recall some definitions and results of lattice theory which will be used in the sequel. Let $(P, \leq)$ be a partially ordered set and $X \subseteq P$. $p \in X$ is a *minimal* element of X if there is no $x \in X$ such that $x < p$. The *greatest* element of X is an element $g \in X$ such that $x \leq g$ for all $x \in X$. A *lattice* is a poset $L = (L, \leq)$ any two of whose elements have a g.l.b. or "meet" denoted by $x \wedge y$, and a l.u.b. or "join" denoted by $x \vee y$. A lattice L is *complete* when each of its subset T has a l.u.b. and a g.l.b. in L. An element a of a lattice L is *join-irreducible* if $x \vee y = a$ implies $x = a$ or $y = a$. A lattice is *Brouwerian* if for any pair of elements $a, b \in L$, the greatest element $x \in L$, denoted by $a\alpha b$, satisfying the inequality $a \wedge x \leq b$ exists. It is easy to verify the following properties: for any a, b, c in a complete Brouwerian lattice L, $b \leq a\alpha(a\wedge b)$, $a\alpha(b\wedge c) = (a\alpha b)\wedge(a\alpha c)$, $a\wedge(a\alpha b) = a\wedge b$. It is well known that any Brouwerian lattice L is distributive, and a complete lattice L is Brouwerian if and only if (iff)

$$x \wedge (\bigvee_{i\in I} y_i) = \bigvee_{i\in I} (x \wedge y_i)$$

for any $x \in L$ and any family of elements $\{y_i \in L : i \in I\}$. Let p be a join-irreducible element of a distributive lattice L. Then $p \leq \bigvee_{i=1}^{k} x_i$ iff $p \leq x_i$ for some x_i. Some other definitions and results of lattice theory which we do not list here are from [1, 4].

2 The sets of solutions for (1)

Throughout the paper, L is always assumed to be a complete Brouwerian lattice with universal bounds 0 and 1, I and J are infinite index sets, unless otherwise specified. Denote $X^* = (x^*_j)^T_{j\in J}$ a greatest solution, $\underline{k} = \{1, 2, \cdots, k\}$ for any positive integer k and $A - B = \{x \in A : x \notin B\}$, where A and B are two (crisp) sets.

It is trivial that $a \wedge x = b$ (resp. $a \vee x = b$) is solvable iff $a \geq b$ (resp. $a \leq b$). Denote $[a]_b$ (resp. $\overline{[a]}_b$) its solution set. For the solvable case, if L is a distributive lattice then $[a]_b = [b) - \bigcup_{b<d\leq a} [d)$ and $\overline{[a]}_b = (b] - \bigcup_{a\leq d<a} (d]$ (see [36]); if L is a complete Brouwerian lattice then $[a]_b = [b, a\alpha b]$ and $\overline{[a]}_b = (b] - \bigcup_{a\leq d<a} (d]$, where $[a) = \{x \in L : x \geq a\}$ and $(a] = \{x \in L : x \leq a\}$ (see [12]). In what follows, for any $i \in I$, denote

$$\mathscr{R}^i = \{R^i = (q_{ij})^T_{j\in J} : b_i = \bigvee_{j\in J} q_{ij}, q_{ij} \leq a_{ij}, \forall j \in J\},$$

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$\mathcal{X}^i$ the solution set of $\bigvee_{j\in J}(a_{ij}\wedge x_j)=b_i$ and $\mathcal{Q}=\{Q=(q_{ij})_{I\times J}:\bigvee_{j\in J}q_{ij}=b_i,\forall i\in I;\bigcap_{i\in I}[a_{ij}]_{q_{ij}}\neq\emptyset,\forall j\in J\}$. Then we can verify the following criteria of solvability easily.

Theorem 2.1 (Di Nola [9]) *$\mathscr{X}_1\neq\emptyset$ iff $\mathcal{Q}\neq\emptyset$.*

Though we omit its proof, it should be pointed out that the proof of Theorem 2.1 suggests a way to build other elements, different from the greatest element $X^*=(\bigwedge_{i\in I}(a_{ij}\alpha b_i))^T_{j\in J}$, of $\mathscr{X}_1$. What is more, Theorem 2.1 holds when L is a complete lattice.

Note that for any $i\in I$, the i-th row of any $Q\in\mathcal{Q}$ forms a decomposition of the i-th component of B. Moreover, $b_i=\bigvee_{j\in J}q_{ij}$ and $q_{ij}\leq a_{ij}$, $\forall j\in J$, therefore $\bigvee_{j\in J}(a_{ij}\wedge q_{ij})=\bigvee_{j\in J}q_{ij}=b_i$, $\forall i\in I$, thus the transpose of the i-th row of Q solves the i-th equation in (1), i.e. $\mathscr{R}^i\subseteq\mathcal{X}^i$, $\forall i\in I$.

Let $\mathcal{Q}_j=\bigcap_{i\in I}[a_{ij}]_{q_{ij}}$ for any $j\in J$. Then the following theorem is true.

Theorem 2.2 (Di Nola [9]) *Let $\mathscr{X}_1\neq\emptyset$. Then for any $j\in J$, $\mathcal{Q}_j=[\bigvee_{i\in I}q_{ij},\bigwedge_{i\in I}(a_{ij}\alpha q_{ij})]$, where $(q_{ij})_{I\times J}\in\mathcal{Q}$. Further, put $R=(r_j)^T_{j\in J}$ with $r_j\in\mathcal{Q}_j$ for all $j\in J$, then $R\in\mathscr{X}_1$.*

Note that Theorem 2.2 gives the solution set of (1) when $|J|=1$, i.e. $[\bigvee_{i\in I}b_i,\bigwedge_{i\in I}(a_i\alpha b_i)]$ is the solution set of the system of equations $a_i\wedge x=b_i$, $i\in I$. Then the following equivalent conditions is straightforward.

Corollary 2.1 (Han [12]) *For the system of equations $a_i\wedge x=b_i$, $i\in\underline{m}$ ($m\geq 2$), the following conditions are equivalent:*
(1) The system of equations $a_i\wedge x=b_i$, $i\in\underline{m}$ is solvable;
(2) $a_i\geq b_i$ and $a_j\wedge b_i=b_j\wedge b_i$ for all $i,j\in\underline{m}$;
(3) $\bigvee_{i=1}^m b_i$ is its solution;
(4) $\bigwedge_{i=1}^m(a_i\alpha b_i)$ is its solution.
Further, the solution set is $[\bigvee_{i=1}^m b_i,\bigwedge_{i=1}^m(a_i\alpha b_i)]$.

Remark 2.1 Conditions (1), (2) and (3) in Corollary 2.1 are also equivalent when the system is assigned on distributive lattices.

By Remark 2.1, the following corollary is obvious when L is a distributive lattice.

Corollary 2.2 (Zhang [35]) *The system of equations $a_i\wedge x=b_i$, $i\in\underline{m}$ is solvable iff $\bigvee_{i=1}^m b_i$ is its solution. And the solution set is $[\bigvee_{i=1}^m a_i]_{\bigvee_{i=1}^m b_i}$.*

Combining Corollaries 2.1 and 2.2, we get a property of α operation as follows:

Proposition 2.1 *If $a_i\geq b_i$ and $a_j\wedge b_i=b_j\wedge b_i$ for all $i,j\in\underline{m}$, then $(\bigvee_{i=1}^m a_i)\alpha(\bigvee_{i=1}^m b_i)=\bigwedge_{i=1}^m(a_i\alpha b_i)$.*

Proof. If $a_i\geq b_i$ and $a_j\wedge b_i=b_j\wedge b_i$ for all $i,j\in\underline{m}$, then $a_i\wedge x=b_i$, $i\in\underline{m}$ is solvable and its solution set is $[\bigvee_{i=1}^m b_i,\bigwedge_{i=1}^m(a_i\alpha b_i)]$ by Corollary 2.1. Therefore from Corollary 2.2, we have $[\bigvee_{i=1}^m b_i,(\bigvee_{i=1}^m a_i)\alpha(\bigvee_{i=1}^m b_i)]=[\bigvee_{i=1}^m a_i]_{\bigvee_{i=1}^m b_i}=[\bigvee_{i=1}^m b_i,\bigwedge_{i=1}^m(a_i\alpha b_i)]$. Thus $(\bigvee_{i=1}^m a_i)\alpha(\bigvee_{i=1}^m b_i)=\bigwedge_{i=1}^m(a_i\alpha b_i)$. □

Notice that, in general, $(\bigvee_{i=1}^m a_i)\alpha(\bigvee_{i=1}^m b_i)=\bigwedge_{i=1}^m(a_i\alpha b_i)$ does not always hold.

Example 2.1 Let N be the set of nonnegative integers. We define $a\wedge b=$ l.c.m.$\{a,b\}$, $a\vee b=$ g.c.d.$\{a,b\}$, $a\leq b$ iff a is multiple of b, where $a,b\in N$ and l.c.m. (resp. g.c.d.) stands for the smallest (resp. greatest) common multiple (resp. divisor) between a and b. Then $L=(N,\wedge,\vee,\leq)$ is a complete Brouwerian lattice with operator "α" given by $a\alpha b=$ g.c.d.$\{x\in N:a\wedge x\leq b\}$ for any $a,b\in N$. $(2\vee 4)\alpha(3\vee 5)=2\alpha 1=1$, but $(2\alpha 3)\wedge(4\alpha 5)=3\wedge 5=15$.

When L is a distributive lattice, $I=\underline{m}$ and $J=\underline{n}$, the next two corollaries are straightforward by Theorem 2.1 and Corollary 2.2.

Corollary 2.3 (Zhang [35]) *Equation (1) is solvable iff there exists a matrix $Q=(q_{ij})_{m\times n}$ such that $\bigvee_{j=1}^n q_{ij}=b_i$, and $a_{ij}\wedge(\bigvee_{k=1}^m q_{kj})=q_{ij}$, $i\in\underline{m}$ and $j\in\underline{n}$. Denote (Q) the set of such Q. Then $\mathscr{X}_1=\bigcup_{Q\in(Q)}H_Q$, where $H_Q=\{(r_j)^T_{j\in\underline{n}}:r_j\in[\bigvee_{i=1}^m a_{ij}]_{\bigvee_{i=1}^m q_{ij}},\forall j\in\underline{n}\}$.*

Corollary 2.4 (Zhang [36]) *For given $B'=(b'_i)^T_{i\in\underline{m}}$, $A'=(a'_i)_{i\in\underline{n}}$ and B, there exists X such that $B'\odot A'\odot X=B$ iff $\bigvee_{j=1}^n a'_j\geq\bigvee_{i=1}^m b_i$ and $b'_k\wedge(\bigvee_{i=1}^m b_i)=b_k$, $\forall k\in\underline{m}$.*

Proof. If there exists $X=(x_j)^T_{j\in\underline{n}}$ such that $B'\odot A'\odot X=B$, then $\bigvee_{j=1}^n(a'_j\wedge x_j)$ is a solution of $b'_i\wedge x=b_i$, $i\in\underline{m}$, thus $\bigvee_{j=1}^n a'_j\geq\bigvee_{j=1}^n(a'_j\wedge x_j)\geq\bigvee_{i=1}^m b_i$. Further, $b'_k\wedge(\bigvee_{i=1}^m b_i)=b_k$, $\forall k\in\underline{m}$ by Corollary 2.2.

Conversely, if $\bigvee_{j=1}^n a'_j\geq\bigvee_{i=1}^m b_i$, then $\bigvee_{j=1}^n[a'_j\wedge(\bigvee_{i=1}^m b_i)]=(\bigvee_{j=1}^n a'_j)\wedge(\bigvee_{i=1}^m b_i)=\bigvee_{i=1}^m b_i$, therefore $X=(x_j)^T_{j\in\underline{n}}$ with $x_j=\bigvee_{i=1}^m b_i$ satisfies $B'\odot A'\odot X=B$ since $b'_k\wedge(\bigvee_{i=1}^m b_i)=b_k$, $\forall k\in\underline{m}$. □

Note that Corollary 2.3 was generalized in [37] to fuzzy relation equations with sup-conjunctor composition.

From Theorem 2.2 we know that $R=(r_j)^T_{j\in J}\in\mathscr{X}_1$ with $r_j\in[\bigvee_{i\in I}q_{ij},\bigwedge_{i\in I}(a_{ij}\alpha q_{ij})]$ for all $j\in J$. Particularly, take $R=(\bigvee_{i\in I}q_{ij})^T_{j\in J}$ with $(q_{ij})_{I\times J}\in\mathcal{Q}$ and $\mathscr{R}=\{R=(r_j)^T_{j\in J}:r_j=\bigvee_{i\in I}q_{ij},\forall j\in J,(q_{ij})_{I\times J}\in\mathcal{Q}\}$. It is easy

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to verify $\mathscr{R} \subseteq \mathscr{X}_1$. For any $X = (x_j)^T_{i\in J} \in \mathscr{X}_1$, we have $R^i = (q_{ij})^T_{j\in J} = (a_{ij} \wedge x_j)^T_{j\in J} \in \mathscr{R}^i, \forall i \in I$. Therefore, $\mathscr{R}$ possesses similar properties as $\mathscr{X}_1$. For instance, we have:

Proposition 2.2 *If $R^1 = (r^1_j)^T_{j\in J}$, $R^2 = (r^2_j)^T_{j\in J} \in \mathscr{R}$ and $R = (r_j)^T_{j\in J}$ such that $R^1 \leq R \leq R^2$, then $R \in \mathscr{R}$.*

Proposition 2.3 *If $R^1 = (r^1_j)^T_{j\in J}$, $R^2 = (r^2_j)^T_{j\in J} \in \mathscr{R}$, then $R^1 \vee R^2 \in \mathscr{R}$.*

Proposition 2.4 *$\mathscr{R} \neq \emptyset$ iff $R^* = ((\bigvee_{i\in I} a_{ij}) \wedge [\bigwedge_{i\in I} (a_{ij}\alpha b_i)])^T_{j\in J} \in \mathscr{R}$. Further, R^* is the greatest element of $\mathscr{R}$.*

Note that, from Proposition 2.4 we know if $|\mathscr{X}_1| = 1$ then $\bigwedge_{i\in I} (a_{ij}\alpha b_i) \leq \bigvee_{i\in I} a_{ij}$ for all $j \in J$, since $R^* \in \mathscr{R} \subseteq \mathscr{X}_1$ must be equal to the greatest element of $\mathscr{X}_1$ (otherwise, $|\mathscr{X}_1| > 1$). By the definition of $\mathscr{R}$, we can build an element of $\mathscr{X}_1$ from $\mathscr{R}^i, i \in I$, which may be different from the greatest element in $\mathscr{X}_1$.

Example 2.2 Let L be the lattice considered in Example 2.1. Consider the fuzzy relational equation

$$\begin{cases} (3 \wedge x_1) \vee (7 \wedge x_2) = 3, \\ (6 \wedge x_1) \vee (8 \wedge x_2) = 12. \end{cases} \tag{3}$$

We can see $(12, 21)^T \in \mathscr{R}^1$, $(12, 24)^T \in \mathscr{R}^2$, i.e. $12\vee 21 = 3$, $12 \vee 24 = 12$. And further, $4 \in [3]_{12} \cap [6]_{12}$, $3 \in [7]_{21} \cap [8]_{24}$, then $(12 \vee 12, 21 \vee 24)^T = (12, 3)^T \in \mathscr{R} \subseteq \mathscr{X}_1$, which is different from the the greatest element $(4, 3)^T$.

Let $\mathscr{R} \neq \emptyset$. From the definition of $\mathscr{R}$, we know that R is an element of $\mathscr{R}$ if and only if there exist $R^i = (q_{ij})^T_{j\in J} \in \mathscr{R}^i$, $\forall i \in I$ such that $R = \bigvee_{i\in I} R^i$ and $\bigcap_{i\in I} [a_{ij}]_{q_{ij}} \neq \emptyset, \forall j \in J$. This result has a generalized version considered over semilinear spaces (see [10, 21, 22] for details) as follows:

Theorem 2.3 (Nosková [16]) *Let (1) be solvable. Then X is a solution to (1) iff there exist $X_i \in \mathcal{X}^i$, $i \in \underline{m}$, such that $X = \bigvee_{i=1}^{m} X_i$ and $\bigvee_{i=1}^{m} X_i \leq X^*$.*

For every $i \in I$, denote $\mathcal{S}^i = \bigcup_{R^i\in\mathscr{R}^i} S_{R^i}$, where $R^i = (q_{ij})^T_{j\in J} \in \mathscr{R}^i$ and $S_{R^i} = \{S = (s_j)^T_{j\in J} : s_j \in [a_{ij}]_{q_{ij}}, \forall j \in J\}$. Then $\mathscr{R}^i \subseteq \mathcal{S}^i$ since $q_{ij} \in [a_{ij}]_{q_{ij}}$. Further,

Theorem 2.4 *$\mathcal{S}^i = \mathcal{X}^i, \forall i \in I$.*

Proof. For any $i \in I$, if $S = (s_j)^T_{j\in J} \in \mathcal{S}^i$, then there exists $R^i = (q_{ij})^T_{j\in J} \in \mathscr{R}^i$ such that $S \in S_{R^i}$, i.e. $s_j \in [a_{ij}]_{q_{ij}}, \forall j \in J$. Thus $\bigvee_{j\in J} (a_{ij} \wedge s_j) = \bigvee_{j\in J} q_{ij} = b_i$. Therefore $S \in \mathcal{X}^i$, i.e. $\mathcal{S}^i \subseteq \mathcal{X}^i$. Vice versa, for any $i \in I$, if $X = (x_j)^T_{j\in J} \in \mathcal{X}^i$, then $\bigvee_{j\in J} (a_{ij} \wedge x_j) = b_i$. Let $R^i = (a_{ij} \wedge x_j)^T_{j\in J}$, i.e. $R^i \in \mathscr{R}^i$. Then $X \in S_{R^i} \subseteq \mathcal{S}^i$ since $x_j \in [a_{ij}]_{a_{ij}\wedge x_j}, \forall j \in J$. Thus $\mathcal{X}^i \subseteq \mathcal{S}^i$, together with $\mathcal{S}^i \subseteq \mathcal{X}^i$ we conclude $\mathcal{S}^i = \mathcal{X}^i$. □

Since $\mathscr{X}_1 = \bigcap_{i\in I} \mathcal{X}^i$, then the following theorem is straightforward by Theorem 2.4.

Theorem 2.5 $\mathscr{X}_1 = \bigcap_{i\in I} \bigcup_{R^i\in\mathscr{R}^i} S_{R^i}$.

From Theorems 2.4 and 2.5, we know that it is important to find all elements of $\mathscr{R}^i, i \in I$, in order to determine $\mathscr{X}_1$.

In the sequel, we will consider a method to determine all elements of $\mathscr{R}^i, i \in I$ then describe the solution set of (1).

Theorem 2.6 *For any $i \in I$, if $\mathscr{R}^i \neq \emptyset$ and $R^* = (q^*_{ij})^T_{j\in J} = (a_{ij} \wedge b_i)^T_{j\in J}$ is the greatest element of $\mathscr{R}^i$, then every element $R = (q_{ij})^T_{j\in J}$ of $\mathscr{R}^i$ is determined by*

$$q_{ij} \in \overline{[\bigvee_{k\in J, k\neq j} q^j_{ik}]_{b_i}} \bigcap (a_{ij}],$$

where

$$q^j_{ik} = \begin{cases} q_{ik}, & k \in D, \\ q^*_{ik}, & k \in J - D \end{cases}$$

and $D = \{j \in J : q_{ij}$ has been determined $\}$.

Proof. For any $R = (q_{ij})^T_{j\in J} \in \mathscr{R}^i$ and $j \in J$, $b_i = \bigvee_{k\in J} q^*_{ik} \geq q_{ij} \vee (\bigvee_{k\in J,k\neq j} q^j_{ik}) \geq q_{ij} \vee (\bigvee_{k\in J,k\neq j} q_{ik}) = b_i$, then $q_{ij} \in \overline{[\bigvee_{k\in J,k\neq j} q^j_{ik}]_{b_i}} \bigcap (a_{ij}]$ since $q_{ij} \leq a_{ij}, \forall j \in J$ by the definition of $\mathscr{R}^i$. Vice versa is straightforward by the definition of q_{ij} and q^j_{ik}. □

By Theorems 2.5 and 2.6 the following results is obvious.

Corollary 2.5 *For any $i \in I$, if $R^* = (q^*_{ij})^T_{j\in J} = (a_{ij} \wedge b_i)^T_{j\in J}$ is the greatest element of $\mathscr{R}^i$, then every element $X = (x_j)^T_{j\in J}$ of $\mathscr{X}_1$ is determined by*

$$x_j \in \bigcap_{i\in I} \bigcup_{q_{ij}\in\overline{[\bigvee_{k\in J,k\neq j} q^j_{ik}]_{b_i}} \bigcap (a_{ij}]} [a_{ij}]_{q_{ij}},$$

where

$$q^j_{ik} = \begin{cases} q_{ik}, & k \in D, \\ q^*_{ik}, & k \in J - D \end{cases}$$

and $D = \{j \in J : q_{ij}$ has been determined $\}$.

Note that Corollary 2.5 generalizes Theorem 3.2 in [12].

As for (2), the corresponding $\mathscr{R}$ mentioned before is nothing but

$$\mathscr{R}_2 = \{R = (r_j)^T_{j\in J} : b = \bigvee_{j\in J} r_j, r_j \leq a_j, \forall j \in J\}.$$

When $J = \underline{n}$, let $b = s_1 = r_1 \vee s_2, s_2 = r_2 \vee s_3, \cdots, s_{n-1} = r_{n-1} \vee s_n, s_n = r_n$. Therefore, $r_1 \vee \cdots \vee r_n = b$ and the following theorem when (2) is assigned on distributive lattices is straightforward, since Theorem 2.4 also holds in distributive lattices in finite case.

Theorem 2.7 (Zhang [35]) *$\mathscr{R}_2 = \bigcup_{b=s_1\geq s_2\geq\cdots\geq s_n} \{R = (r_j)^T_{j\in\underline{n}} : r_j \in \overline{[s_{j+1}]_{s_j}} \bigcap (a_j], \forall j \in \underline{n-1}, r_n = s_n\}$. And $\mathscr{X}_2 = \bigcup_{b=s_1\geq s_2\geq\cdots\geq s_n} \{X = (x_j)^T_{j\in\underline{n}} : x_j \in [a_j]_{r_j}, \forall j \in \underline{n}, r_j \in \overline{[s_{j+1}]_{s_j}} \bigcap (a_j], \forall j \in \underline{n-1}, r_n = s_n\}$.*

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3 Conditions for existence of minimal solutions to (1)

As it was mentioned in [8] that the determination of minimal elements in $\mathscr{X}_1$, i.e. minimal solutions to (1), when (1) is assigned on complete Brouwerian lattices remains open in finite case as well as in infinite case. In this section we will consider the determination of minimal solutions.

First, we have:

Theorem 3.1 (Di Nola [9]) *If $\mathscr{R}$ has minimal elements then these elements are minimal in $\mathscr{X}_1$, and vice versa.*

As a comment to Theorem 3.1 we have to say we do not know if $\mathscr{X}_1$ (or $\mathscr{R}$) has or has not minimal elements. However, if one suppose the existence of minimal elements in $\mathscr{X}_1$, then these elements must be sought among the minimal elements of $\mathscr{R}$.

Minimal solutions do not always exist, here is an example borrowed from [29].

Example 3.1 Let L be the lattice considered in Example 2.1. If $A = (3, 7)$ and $b = 2$, then (2) has a solution $X = (2, 2)^T$ but no minimal solution since for any $X = (x_1, x_2)^T \in \mathscr{X}_2$, $X' = (2x_1, x_2)^T$ is also an element of $\mathscr{X}_2$ and $X' \leq X$ but $X' \neq X$.

Let $\mathscr{X}_1 \neq \emptyset$ and $\gamma_{j_0}(X) = \{l \in L : (a_{ij_0} \wedge l) \vee [\bigvee_{j\in J, j\neq j_0} (a_{ij} \wedge x_j)] = b_i, \forall i \in I\}$ for every $X = (x_j)^T_{j\in J} \in \mathscr{X}_1$ and $j_0 \in J$. Then:

Theorem 3.2 *For any $X = (x_j)^T_{j\in J} \in \mathscr{X}_1$, if $X' = (x'_j)^T_{j\in J} \in \mathscr{X}_1$ and $X' \geq X$ then $x_j \in \gamma_j(X')$ for all $j \in J$.*

Proof. For any $X = (x_j)^T_{j\in J} \in \mathscr{X}_1$ and $j_0 \in J$, put $\overline{X} = (\overline{x}_j)^T_{j\in J}$ with

$$\overline{x}_j = \begin{cases} x_{j_0}, & j = j_0, \\ x'_j, & j \neq j_0. \end{cases}$$

Then $X \leq \overline{X} \leq X'$, and $\overline{X} \in \mathscr{X}_1$. Therefore, $x_{j_0} \in \gamma_{j_0}(X')$. □

The following theorem gives a criterion to determine when a given solution is minimal:

Theorem 3.3 (Di Nola [9]) *$X = (x_j)^T_{j\in J}$ is a minimal element of $\mathscr{X}_1$ iff $x_{j_0} = min\ \gamma_{j_0}(X)$ for every $j_0 \in J$.*

For a given element $R = (r_j)^T_{j\in J} = (\bigvee_{i\in I} q_{ij})^T_{j\in J}$ in $\mathscr{R}$ and each $j_0 \in J$, denote $\overline{\gamma}_{j_0}(R) = \{t \in L : t = \bigvee_{i\in I} t_{ij_0}, (\bigvee_{j\in J, j\neq j_0} q_{ij}) \vee t_{ij_0} = b_i, \forall i \in I\}$, then from Theorems 3.2 and 3.3 the following two theorems are obvious.

Theorem 3.4 *$R = (r_j)^T_{j\in J}$ is a minimal element of $\mathscr{R}$ iff $r_{j_0} = min\ \overline{\gamma}_{j_0}(R)$ for all $j_0 \in J$.*

Theorem 3.5 *For any $R = (r_j)^T_{j\in J} \in \mathscr{R}$, if $R' = (r'_j)^T_{j\in J} \in \mathscr{R}$ and $R' \geq R$ then $r_j \in \overline{\gamma}_j(R')$ for all $j \in J$.*

As an immediate consequence of Theorem 3.3, the following theorem on the unicity of solution is obvious.

Theorem 3.6 (Di Nola [9]) *$|\mathscr{X}_1| = 1$ iff for every $j_0 \in J$, $x^*_{j_0} = \bigwedge_{i\in I} (a_{ij_0} \alpha b_i) = min\ \gamma_{j_0}(X^*)$.*

In fact, if $\mathscr{X}_1 \neq \emptyset$, then from Theorem 2.2 we have that $|\mathscr{X}_1| = 1$ if and only if $\bigvee_{i\in I} q_{ij} = \bigwedge_{i\in I} (a_{ij} \alpha q_{ij})$ for any $j \in J$ and $(q_{ij})_{I\times J} \in \mathscr{Q}$ (see also Theorem 6 in [25]).

By Theorem 3.1, the following theorem is straightforward.

Theorem 3.7 (Wang [30]) *If $X_* = (x_{j_*})^T_{j\in J}$ is a minimal element of $\mathscr{X}_2$, then $a_j \wedge x_{j_*} = x_{j_*}$ for all $j \in J$, i.e. $b = \bigvee_{j\in J} x_{j_*}$ and $x_{j_*} \leq a_j, \forall j \in J$.*

Though minimal solutions do not always exist, when adding some assumptions to B, we get some interesting results. Here is a sufficient condition for existence of a minimal solution to (1) when I is finite and J is infinite.

Theorem 3.8 (Wang [28]) *If every component of B is a compact element with an irredundant finite join-decomposition, then for each $X \in \mathscr{X}_1$ there exists a minimal element $X_* \in \mathscr{X}_1$ such that $X_* \leq X$.*

Thus the following theorem follows from Theorems 3.1 and 3.8 easily.

Theorem 3.9 *Suppose that each component of B is compact and has an irredundant finite join-decomposition. Then R is minimal in $\overline{\mathscr{R}}$ iff it is minimal in $\mathscr{X}_1$, where $\overline{\mathscr{R}} = \{R : R = \bigvee_{i\in I} R^i, R^i = (q_{ij})^T_{j\in J}$ is minimal in $\mathscr{R}^i, \forall i \in I; \bigcap_{i\in I} [a_{ij}]_{q_{ij}} \neq \emptyset, \forall j \in J\}$.*

Note that Theorem 3.8 was generalized in [31] to fuzzy relation equations with sup-conjunctor composition. With Theorem 3.9 in hands, it is easy to see that if I is finite and every component of B is compact and has an irredundant finite join-decomposition, then $\mathscr{X}_1$ is completely determined by the greatest solution and all minimal solutions. Therefore, Theorem 3.9 generalizes Theorem 3.11 in [8] and the corresponding result in [13] considered over $[0, 1]$. Another generalized version of Theorem 3.11 in [8] is Theorem 4 in [16] considered fuzzy relation equations over semi-linear spaces. When J is also finite, the assumption of compact in Theorem 3.8 can be removed (see Theorem 7.1 in [29]). Further, the following theorem gives the number of minimal solutions of (2) when both I and J are finite.

Theorem 3.10 (Wang [28]) *If $\mathscr{X}_2 \neq \emptyset$ and b has an irredundant finite join-decomposition $\bigvee_{i=1}^{k} p_i$, then the number of minimal solutions of (2) is $\prod_{i\in \underline{k}} |G(p_i)|$, where $G(p_i) = \{j \in \underline{n} : a_j \geq p_i\}$. Further, all minimal solutions $X = (x_j)^T_{j\in J}$ are determined by $x_j = \bigvee_{f_X(i)=j} p_i$ for any mapping $f_X \in \prod_{i\in \underline{k}} G(p_i)$.*

Let both I and J be finite. If for any $i \in I$, b_i has an irredundant finite join-decomposition $\bigvee_{t=1}^{k_i} p_{it}$, then for any $i \in I, t_i \in \underline{k_i}$ we define $G(p_{it_i}) = \{j \in J : a_{ij} \geq p_{it_i}\}$ and $F_i = \{f_i : f_i \in \prod_{t_i\in \underline{k_i}} G(p_{it_i}), \bigvee_{f_i(t_i)=j} p_{it_i} \leq \bigwedge_{i\in I} (a_{ij} \alpha b_i), \forall j \in J\}$. Then:

Theorem 3.11 *Let $\mathscr{X}_1 \neq \emptyset$. Then $|\mathscr{X}_1| = 1$ iff for any $j \in J$ and $f_i \in F_i$, $\bigvee_{i\in I} \bigvee_{f_i(t_i)=j} p_{it_i} = \bigwedge_{i\in I} (a_{ij} \alpha b_i)$.*

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Proof. Note that $X^* = (\bigwedge_{i\in I}(a_{ij}\alpha b_i))^T_{j\in J} \in \mathscr{X}_1$ since $\mathscr{X}_1 \neq \emptyset$. For any $i \in I$ and $f_i \in F_i$, from Theorem 3.10 and the definition of F_i we know that $(\bigvee_{f_i(t_i)=j} p_{it_i})^T_{j\in J} \leq X^*$ and $(\bigvee_{f_i(t_i)=j} p_{it_i})^T_{j\in J}$ is minimal in $\mathcal{X}^i$. Thus $\{X = (x_j)^T_{j\in J} : x_j = \bigvee_{i\in I}\bigvee_{f_i(t_i)=j} p_{it_i}, f_i \in F_i\}$ contains all minimal elements of $\mathscr{X}_1$ from Theorem 3.9. Therefore, the thesis is straightforward. □

Note that Theorem 3 in [7] that considers the unique solvability of fuzzy relation equations over linear lattices and it is just a special case of Theorem 3.11.

When $|I| = 1$ and J is infinite, a necessary and sufficient condition for existence of a minimal solution is as follows:

Theorem 3.12 (Wang [30]) *Let $\mathscr{X}_2 \neq \emptyset$. Then for each $X \in \mathscr{X}_2$ there exists a minimal element X_* of $\mathscr{X}_2$ such that $X_* \leq X$ iff there is a subset B of L with B satisfying:*
(i) $\bigvee B = b$;
(ii) For each $p \in B$, if $p \neq 0$ then $b \neq \bigvee(B - \{p\})$;
(iii) For each $X = (x_j)^T_{j\in J} \in \mathscr{X}_2$ and each $p \in B$ there is an index $k \in J$ such that $p \leq a_k \wedge x_k$.

If I is finite, then from Theorem 3.12 we have

Theorem 3.13 (Wang [30]) *If $\mathscr{X}_1 \neq \emptyset$ and every component b_i, $i \in I$, of B is compact and for each b_i, $i \in I$, there exists a subset B_i of L such that:*
(i) $\bigvee B_i = b_i$;
(ii) For each $p_{it} \in B_i$, if $p_{it} \neq 0$ then $b_i \neq \bigvee(B_i - \{p_{it}\})$;
(iii) For each $X = (x_j)^T_{j\in J} \in \mathcal{X}^i$ and each $p_{it} \in B_i$ there is an index $k \in J$ such that $p_{it} \leq a_{ik} \wedge x_k$;
(iv) For each $p \in \bigcup_{i\in I} B_i$, if $p \neq 0$ then there is no subset Q of $\bigcup_{i\in I} B_i$ such that $p \leq \bigvee(Q - \{p\})$.
Then for each $X \in \mathscr{X}_1$, there exists a minimal element X_* of $\mathscr{X}_1$ such that $X_* \leq X$.

Notice that Theorem 3.8 is a special case of Theorem 3.13. In the following, let $G(b) = \{j \in J : a_j \geq b\}$. Then

Theorem 3.14 *Let $\mathscr{X}_2 \neq \emptyset$. Then there exists a minimal element in $\mathscr{X}_2$ iff either $G(b) \neq \emptyset$ or there is a subset B of L such that:*
(i) $b = \bigvee B$;
(ii) For every $p \in B$, there exists an index $j \in J$ such that $p \leq a_j$;
(iii) For every $p \in B, q \in L$, if $q < p$ then $b \neq [\bigvee(B - \{p\})] \bigvee q$.

Proof. Let $X_* = (x_{*j})^T_{j\in J}$ be minimal in $\mathscr{X}_2$. If $G(b) = \emptyset$ then from Theorem 3.7 $b = \bigvee_{j\in J} x_{*j}$, put $B = \{x_{*j} : x_{*j} \neq 0, j \in J\}$, then B satisfies the conditions (i), (ii) and (iii).

Conversely, if $G(b) \neq \emptyset$ then it is easy to see that there exists a minimal element in $\mathscr{X}_2$. Now suppose that there exists a subset B of L satisfying (i), (ii) and (iii), then we can construct a family of subsets $A_j, j \in J$ of B such that $\bigcup_{j\in J} A_j = B$ and $A_i \bigcap A_k = \emptyset$ when $i \neq k$ and $i, k \in J$. Define $X = (x_j)^T_{j\in J}$ with

$$x_j = \begin{cases} \bigvee_{p\in A_j} p, & A_j \neq \emptyset, \\ 0, & A_j = \emptyset. \end{cases}$$

Then $\bigvee_{j\in J}(a_j \wedge x_j) = \bigvee_{j\in J, A_j\neq\emptyset} x_j = \bigvee(\bigcup_{j\in J} A_j) = \bigvee B = b$, i.e. $X \in \mathscr{X}_2$. Let $Y = (y_j)^T_{j\in J} \in \mathscr{X}_2$ be such that $Y \leq X$. If $x_j = 0$ then $y_j = 0$. If $x_j \neq 0$ then $A_j \neq \emptyset$, therefore $y_j \leq x_j = \bigvee_{p\in A_j} p \leq a_j$ and $b = \bigvee_{j\in J}(a_j \wedge y_j) = \bigvee_{j\in J} y_j$. One can verify that for any $j \in J$ if $A_j \neq \emptyset$ then $y_j = x_j$. Indeed, if there exists an index $j_0 \in J$ such that $A_{j_0} \neq \emptyset$ but $y_{j_0} < x_{j_0}$, put $\overline{Y} = (\overline{y}_j)_{j\in J}$ with

$$\overline{y}_j = \begin{cases} y_{j_0}, & j = j_0, \\ x_j, & j \neq j_0. \end{cases}$$

Then $b = \bigvee_{j\in J} y_j \leq \bigvee_{j\in J} \overline{y}_j \leq \bigvee_{j\in J} x_j = b$ which means $(\bigvee_{j\in J, j\neq j_0} x_j) \bigvee y_{j_0} = b$, contradicts to (iii). Thus $Y = X$, i.e. X is minimal in $\mathscr{X}_2$. □

Note that Theorem 3.14 partly answers the open problem for existence of minimal solutions (see Page 46 in [8]).

It should be pointed out that most of the results in last two sections, such as Theorems 2.1, 2.4, 2.5, 3.1, 3.2 and 3.3, are also hold when L is distributive in finite case.

4 Conclusions

In this paper we gave an overview of the known results for fuzzy relation equations over complete Brouwerian lattices. In our opinions, the resolution of fuzzy relation equations over complete Brouwerian lattices can be solved from the viewpoint of decomposition.

References

[1] G. Birkhoff, Lattice theory. Vol. XXV, 3rd Ed., American Mathematical Society Colloquium Publications, Providence, RI, 1979.

[2] M. M. Bourke, D. G. Fisher, Solution algorithms for fuzzy relational equations with max-product composition, Fuzzy Sets and Systems 94 (1998) 61-69.

[3] K. Cechlarova, Unique solvability of max-min fuzzy equations and strong regularity of matrices over fuzzy algebra, Fuzzy Sets and Systems 75 (1995) 165-177.

[4] P. Crawley, R. P. Dilworth, Algebraic theory of lattices, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[5] B. De Baets, Sup-T equations: state of the art, in: O. Kaynak, L. Zadeh, B. Turksen, I. Rudas (Eds.), Computational Intelligence: Soft Computing and Fuzzy-Neural Integration with Applications, NATO ASI Series F, Vol. 162, Springer, Berlin, 1998, pp. 80-93.

[6] B. De Baets, Analytical solution methods for fuzzy relational equations, in: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Vol. 1, Kluwer Academic Publishers, Dordrecht, 2000, pp. 291-340.

[7] A. Di Nola, S. Sessa, Finite fuzzy relation equations with a unique solution in linear lattices, J. Math. Anal. Appl 132 (1988) 39-49.

ISBN: 978-989-95079-6-8

[8] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy relation equations and their applications to knowledge engineering. Kluwer Academic Publishers, Dordrecht, 1989.

[9] A. Di Nola, On solving relational equations in Brouwerian lattices, Fuzzy Sets and Systems 34 (1990) 365-376.

[10] A. Di Nola, A. Lettieri, I. Perfilieva, V. Novák, Algebraic analysis of fuzzy systems, Fuzzy Sets and Systems 158 (2007) 1-22.

[11] S. C. Han, H. X. Li, J. Y. Wang, Resolution of finite fuzzy relation equations based on strong pseudo-t-norms, Appl. Math. Lett. 19 (2006) 752-757.

[12] S. C. Han, H. X. Li, J. Y. Wang, Resolution of matrix equations over arbitrary Brouwerian lattices, Fuzzy Sets and Systems 159 (2008) 40-46.

[13] M. Higashi, G. J. Klir, Resolution of finite fuzzy relation equations, Fuzzy Sets and Systems 13 (1984) 65-82.

[14] H. Imai, M. Miyakoshi and T. Da-te, Some properties of minimal solutions for a fuzzy relation equation, Fuzzy Sets and Systems 90 (1997) 335-340.

[15] H. Imai, K. Kikuchi and M. Miyakoshi, Unattainable solutions of a fuzzy relation equation, Fuzzy Sets and Systems 99 (1998) 193-196.

[16] L. Nosková, I. Perfilieva, System of fuzzy relation equations with sup-* composition in semi-linear spaces: minimal solutions. In Proc. FUZZ-IEEE Conf. on Fuzzy Systems, July 23-36, 2007, London, pp. 1520-1525.

[17] W. Pedrycz, s-t Fuzzy relational equations, Fuzzy Sets and Systems 59 (1993) 189-195.

[18] K. Peeva, Y. Kyosev, Fuzzy relational calculus-theory, applications and software (with CD-ROM), in Series Advances in Fuzzy Systems - Applications and Theory, Vol. 22, World Scientific Publishing Company, 2004.

[19] K. Peeva, Y. Kyosev, Algorithm for solving max-product fuzzy relational equations, Soft Comput 11 (2007) 593-605.

[20] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363-383.

[21] I. Perfilieva, Semi-linear spaces, in: H. Noguchi, H. Ishii, et al.(Eds.), Proc. of Seventh Czech-Japanese Seminar on Data Analysis and Decision Making under Uncertainty, Hyogo, Japan, 2004, pp. 127-130.

[22] I. Perfilieva, Semilinear spaces-basic structures for fuzzy systems, in: Proc. Conf. of IPMU'2006, Paris, France, 2006, pp. 1087-1094.

[23] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of IF-THEN rules, Information Sciences 177 (2007) 3218-3227.

[24] E. Sanchez, Resolution of composite fuzzy relation equations, Inform. and Control 30 (1976) 38-48.

[25] S. Sessa, Finite fuzzy relation equations with unique solution in complete Brouwerian lattices, Fuzzy Sets and Systems 29 (1989) 103-113.

[26] M. Wagenknecht, K. Hartmann, On the existence of minimal solutions for fuzzy equations with tolerances, Fuzzy Sets and Systems 34 (1990) 237-244.

[27] X. P. Wang, Method of solution to fuzzy relation equations in a complete Brouwerian lattice, Fuzzy Sets and Systems 120 (2001) 409-414.

[28] X. P. Wang, Infinite fuzzy relational equations in a complete Brouwerian lattice, Indian J. Pure Appl. Math. 33 (2002) 87-95.

[29] X. P. Wang, Conditions under which a fuzzy relational equation has minimal solutions in a complete Brouwerian lattice, Advances In Mathematics 31 (2002) 220-228 (in Chinese).

[30] X. P. Wang, Infinite fuzzy relational equations on a complete Brouwerian lattice, Fuzzy Sets and Systems 138 (2003) 657-666.

[31] X. P. Wang, Q. Q. Xiong, The solution set of a fuzzy relational equation with sup-conjunctor composition in a complete lattice, Fuzzy Sets and Systems 153 (2005) 249-260.

[32] X. P. Wang, X. B. Qu, Continuous join-irreducible elements and their applications to describing the solution set of fuzzy relational equations, Acta Mathematica Sinica, Chinese Series 49 (2006) 1171-1180 (in Chinese).

[33] X. B. Qu, X. P. Wang, Some properties of infinite fuzzy relational equations on complete Brouwerian lattices, Fuzzy Sets and Systems 158 (2007) 1327-1339.

[34] Z. D. Wang, Y. D. Yu, Pseudo-t-norms and implication operators on a complete Brouwerian lattice, Fuzzy Sets and Systems 132 (2002) 113-124.

[35] K. L. Zhang, matrix equations in distributive lattices, Acta Scientiarum Naturalium Universitatis Intramongolicas 22 (1991) 311-317.

[36] K. L. Zhang, The inverse problem for distributive lattices, Fuzzy Sets and Systems 59 (1993) 109-113.

[37] K. L. Zhang, D. H. Li, L. X. Song, On finite relation equations with sup-conjunctor composition over a complete lattice, Fuzzy Sets and Systems 160 (2009) 119-128.

[38] C. K. Zhao, On matrix equations in a class of complete and completely distributive lattices, Fuzzy Sets and Systems 22 (1987) 303-320.

ISBN: 978-989-95079-6-8