Aggregation operators for conditional crispness

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Abstract— *In axiomatic way, in fuzzy setting, we introduce the measure of crispness and the measure of conditional clearness. For these measures we propose a functional form. Then, for the last measure, we give some aggregation operators.*

Keywords— Aggregation operators, functional equations, fuzzy sets, measures of fuzziness, crispness measures, conditional crispness.

1 Introduction

De Luca and Termini gave the first definition of fuzziness measure [8, 15], based upon a condition of monotonicity, with respect to a antural relation of less or grater fuzziness. In 1975 Mukaidono presented another order, called monotonicity with respect to ambiguity [11]. Later Couso and Gil have modified the definition [7] and Roventa used the cuts of fuzzy sets in order to define this kind of entropy [12, 13, 14]. In [2] the authors, taking into account some consideration due to Wang [18] introduced small variations, leanding to a more exact order relation among fuzzy sets, which represents the idea of less or greater fuzziness.

We know that in $\mathcal{F}$ there are two different partial order: the classical inclusion relation [19] and the so called *fuzziness relation* according to the greater or smaller "proximity" to a crisp set; with respect to the last relation it is possible to define the infimum and the supremum [16]. In [2, 3, 4] we have studied the fuzziness measure and in [16] there are many examples of fuzziness measure also by using Sugeno's integral.

These researches have suggested to introduced, in axiomatic way, the concept of crispness c through a fuzziness measure d.

Then, taking into account the crispness, we give the definition of conditional crispness.

We know that an aggregation operator is a procedure by which a unique value can be associated to the result obtained through different tests or different values of data base. The unique value is kind of aritmetic mean.

In this setting, in this paper, we give the definition of measure of crispness (shortly crispness) through the fuzziness measure and the definition of conditional crispness.

Then, we propose a characterization of classes of aggregation operators for the conditional crispness.

2 Preliminaires

Given an abstract space X and a σ-algebra $\mathcal{C}$ of crisp subsets C of X, let $\mathcal{F}$ be the family of all $\mathcal{C}$-measurable fuzzy sets [18, 19]. With every fuzzy set F a membership function f is associated, $f \to [0, 1]$, which is $\mathcal{C}$-measurable.

Now, we recall the order relation called *fuzziness relation* [2, 3]: F_1 is *less fuzzy* then F_2 ($F_1 \preceq F_2$) if the following condition holds $\forall\, x \in X$:

$$\begin{cases} f_1(x) \le \dfrac{1}{2} \Longrightarrow f_1(x) \le f_2(x) \le \dfrac{1}{2} \\ \\ f_1(x) \ge \dfrac{1}{2} \Longrightarrow \dfrac{1}{2} \le f_2(x) \le f_1(x)\ , \end{cases}$$

which is equivalent to

$$\begin{cases} f_1(x) < \dfrac{1}{2} \Longrightarrow f_1(x) \le f_2(x) \\ \\ f_1(x) > \dfrac{1}{2} \Longrightarrow f_1(x) \ge f_2(x)\ . \end{cases}$$

We remember that the fuzziness relation is different and not comparable with the inclusion relation:
$F_1 \subset F_2 \Longleftrightarrow f_1(x) \le f_2(x)\ \forall\, x \in X.$

For crisp sets the relation $C_1 \preceq C_2$ implies $C_1 = C_2$.

We know [2] that $\left[\dfrac{X}{2}\right]$ and $\left[\dfrac{C}{2}\right]$ are those fuzzy sets, whose membership function are $f(x) = \dfrac{1}{2}, \forall x \in X$, and $f(x) = \dfrac{1}{2}, \forall x \in C$, respectively. Moreover we shall indicate with F^c the complement of F.

Definition 1 *The fuzziness measure is a map $d : \mathcal{F} \to [0, 1]$ such that:*

$[A_1]$ $d(C) = 0\ ,\ \forall\, C \in \mathcal{C}\ ;$

$[A_2]$ $F_1 \preceq F_2 \Longrightarrow d(F_1) \le d(F_2)\ \forall F_1, F_2 \in \mathcal{F};$

$[A_3]$ $d\left(\left[\dfrac{X}{2}\right]\right) = 1\ ;$

$[A_4]$ $d(F) = d(F^c)\ \forall F \in \mathcal{F}.$

Moreover, we have recognized in [2] that to every fuzziness measure d we can associate a fuzzy measure μ on $\mathcal{C}$. In fact, the function $\mu : \mathcal{C} \to [0, 1]$, defined by $\mu(C) = d\left(\left[\dfrac{C}{2}\right]\right)$, is non-decreasing with respect to the inclusion relation and $\mu(\emptyset) = 0$. This measure μ is called by us *fuzzy measure associated to* d. In [2] it is showed a form of this measure by using the Sugeno integral.

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3 Crispness and conditional crispness for fuzzy sets

In this paragraph we introduce the definition of crispness by using the fuzziness measure.

Definition 2 *Given a fuzziness measure* d, *the* crispness *of a fuzzy set* F *is a map* $c : \mathcal{F} \to [0,1]$ *defined by*

$$c(F) = 1 - d(F)\ . \tag{1}$$

Using the properties above $[A_1] - [A_4]$ the crispness (1) enjoys the following:

$[A_1^{'}]$ $c(C) = 1\ ,\ \forall\ C \in \mathcal{C},$ as d(C) = 0;

$[A_2^{'}]$ $F_1 \preceq F_2 \implies c(F_1) \geq c(F_2),$
as $d(F_1) \leq d(F_2)\ \forall F_1, F_2 \in \mathcal{F}$;

$[A_3^{'}]$ $c\left(\left[\frac{X}{2}\right]\right) = 0,$ as $d\left(\left[\frac{X}{2}\right]\right) = 1$;

$[A_4^{'}]$ $c(F) = c(F^c)\ \forall F \in \mathcal{F},$ as d(F) = d(F^c) .

As an extension of crisp setting [9], we assume that two fuzzy sets F and H are *algebrically independent* if $F \cap H \neq \emptyset$.

Definition 3 *Fixed a fuzzy set* $H \in \mathcal{F}$ *and a crispness (1), the* conditional crispness *of any variable set* $F \in \mathcal{F}$ *is a map*

$$c_H(\cdot) : \mathcal{F} \to [0,1]$$

defined by the following axioms:

(i) $F_1 \preceq F_2 \implies c_H(F_1) \geq c_H(F_2), \forall F_1, F_2 \in \mathcal{F}$;

(ii) $c_H\left(\left[\frac{X}{2}\right]\right) = 0,\ \left[\frac{X}{2}\right] \in \mathcal{C},$

(iii) $c_H(F) = c(F), \forall F \in \mathcal{F},$
if F is c-independent by H.

We assume the (iii) as the c-independence i.e.independence with respect to c.

As conseguence, every crisp set C is c-independent from H:
$c_H(C) = c(H),\ \forall C \in \mathcal{C}.$

4 Example

Let F be a fuzzy set composed by a group of old men. Given a fuzziness measure d, it is possible to calculate $d(F)$ and $c(F)$ according with (1). Moreover, we consider another fuzzy set H, whose elements are sick men. For us, the meaning of $c_H(F)$ is that the crispness of F influenced by the sickness of the men in H.

5 A form of measure of conditional crispness

Now, we propose the following form for the conditional crispness measure:

$$c_H(F) = \Phi\Big(c(F), c(H)\Big) \tag{2}$$

where $\Phi : [0,1] \times [0,1] \to [0,1]$. From $[(i), (ii), (iii)]$, we obtain:

$(a) F_1 \preceq F_2 \implies \Phi\Big(c(F_1), c(H)\Big) \geq \Phi\Big(c(F_2), c(H)\Big),$
(b) $\Phi(0, c(H)) = 0,$
(c) $\Phi(1, c(H)) = c(H).$

If we put $x = c(F_1), x' = c(F_2), y = c(H)$, with $, x, x', y \in [0,1]$, we get:

$$\begin{cases} (a')\ \Phi(1,y) = y \\ (b')\ \Phi(0,y) = 0 \\ (c')\ \Phi(x,y) \geq \Phi(x',y) \text{ if } x \geq x'\ . \end{cases}$$

For the system $[(a') - (c')]$ we have the following:

Proposition 1 *A class of solution of the system* $[(a') - (c')]$ *is*

$$\Phi = h^{-1}\Big(h(x) \cdot h(y)\Big) \tag{3}$$

where $h : [0,1] \times [0,1] \to [0,1]$ *is increasing with* $h(0) = 0$ *and* $h(1) = 1$.

Proof. The proof is immediate.

6 Example

If $h(x) = x^n, n \in (0, +\infty)$, then $\Phi(x,y) = x \cdot y$.

7 Aggregation operators for conditional crispness measures

In [17] we have studied the aggregation operators of the general conditional information. We shall use again the same procedure for the characterization of some forms of aggregation operators of the conditional crispness.

Let $\mathcal{I}$ be the family of the conditional crispness measures $c_H(\cdot)$. The aggregation operator $L : \mathcal{I} \to [0, \mathrm{K}],\ 0 < \mathrm{K} < [0, +\infty]$ of $n \in [0, +\infty)$ conditional crispness measures $c_H(F_1), ..., c_H(F_i), ..., c_H(F_n)$, with $F_i \in \mathcal{F}, \mathrm{i} = 1, ..., \mathrm{n}, \mathrm{H} \in \mathcal{F}$ has the following properies as in [10]:

(I) $idempotence : c_H(F_i) = \lambda, \forall i = 1, ..., n \implies$

$$L(\underbrace{\lambda, ..., \lambda}_{n\ times}) = \lambda;$$

(II) $monotonicity : c_H(F_1) \leq c_H(F_1^{'}) \implies$
$L(c_H(F_1), ..., c_H(F_i), ..., c_H(F_n)) \leq$
$L(c_H(F_1^{'}), ..., c_H(F_i), ..., c_H(F_n)),$
$F_1^{'}, F_i \in \mathcal{F}, \mathrm{i} = 1, ..., \mathrm{n};$

(III) $continuity\ from\ below : c_H(F_{1,m}) \nearrow c_H(F_1) \implies$

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$c_H(F_{1,m}),...,c_H(F_i),...,c_H(F_n)) \nearrow c_H(F_1),...,c_H(F_i),...,c_H(F_n))$ $F_1^{'}, F_i \in \mathcal{F}$, i = 1, ..., n.

Putting $c_H(F_i) = x_i, i = 1,...n, c_H(F_1^{'}) = x_1^{'}, c_H(F_{1,m}) = x_{1,m}$, with $x_i, i = 1,...n, x_1^{'}, x_{1,m} \in [0,1]$, we obtain the following system of functional equations:

$$\begin{cases} (I')\ L(\underbrace{\lambda,...,\lambda}_{n\ times}) = \lambda, \\ \\ (II')\ x_1 \leq x_1^{'} \Longrightarrow L(x_1,...,x_n) \leq L(x_1^{'},...x_n), \\ \\ (III')\ x_{1,m} \nearrow x_1 \Longrightarrow L(x_{1,m},...,x_n) \nearrow L(x_1,...x_n). \end{cases}$$

For the solution of the system $[(I') - (III')]$, we propose the following:

Proposition 2 *Two natural solutions of the system* $[(I') - (III')]$ *are*

$$L(x_1,...,x_n) = \bigwedge_{i=1}^{n} x_i$$

and

$$L(x_1,...,x_n) = \bigvee_{i=1}^{n} x_i.$$

Proof. The proof is immediate.

Proposition 3 *A class of solution of the system* $[(I')-(III')]$ *is*

$$L(x_1,...,x_n) = h^{-1}\left(\frac{h(x_1)+...+h(x_n)}{n}\right),$$

where $h : [0,1] \rightarrow [0,K](0 < K < +\infty)$ *is a continuous, strictly increasing function with* $h(0) = 0$ *and* $h(1) = K$.

Proof. The proof is immediate.

8 Remark

If the function h is linear, then the aggregation operator L is the aritmetic mean.

9 Conclusion

First, we have given the definitions of the crispness and the conditional crispness for a fuzzy set.

Second, we have proposed some classes of aggregation operators of the conditional crispness, solving a suitable system of functional equations.

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