Consistency of pair-wise comparison matrix with fuzzy elements

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Abstract—In this paper, consistency of pair-wise comparison matrix with fuzzy elements based on the method of geometric mean is investigated. A consistency index of reciprocal matrix with fuzzy elements is introduced based on a newly designed method of logarithmic least squares for eliciting associated weights. Some basic properties of the index are presented and two simple illustrating examples are supplied.

Keywords—decision making, uncertainty, pair-wise comparison, reciprocal matrix, consistency, triangular fuzzy numbers.

1 Introduction

The classical pair-wise comparison method requires the decision-maker (DM) to express his/her preferences in the form of a precise ratio matrix encoding a valued preference relation. However, it can often be difficult for the DM to express exact estimates of the ratios of importance. Therefore many kinds of methods employing intervals or fuzzy numbers as elements of a pair-wise comparison matrix have been proposed to cope with this problem. This allows for a more flexible specification of pair-wise preference intensities accounting for the incomplete knowledge of the DM, see [4].

In practice, when interval-valued matrices are employed, the DM often gives ranges narrower than his/her actual perception would authorize, because he/she might be afraid of expressing information which is too imprecise. On the other hand, a fuzzy number or fuzzy interval expresses rich information because the DM can provide the core of the fuzzy interval as a rough estimate of his perceived preference and also the support set of the fuzzy interval as the range that the DM believes to surely contain the unknown ratio of relative importance.

In this paper, consistency of pair-wise comparison matrix with fuzzy numbers based on the method of geometric mean is investigated. A consistency index of reciprocal matrix with triangular fuzzy elements is introduced based on a newly designed method of logarithmic least squares for eliciting associated weights with the minimal measure of fuzziness. Here, we accept the reasons discussed in [3] (and in other papers cited there) for using geometric mean instead of Saaty's procedures based on the principal eigenvector of the pair-wise comparison matrix. Our approach is however different to the method used by Van Laarhoven and Pedrycz, see [7]. Defining the consistency index, we use the different approach to Buckley's at al. [2], the consistency index proposed there is based on the Saaty's definition. Moreover, the calculation procedure of the index is rather complex.

2 Consistency of pair-wise comparison matrix

Consider an n×n pair-wise comparison matrix **A** such that

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix}. \tag{1}$$

This matrix is reciprocal, if $a_{ij} = \frac{1}{a_{ji}}$ for each $1 \le i, j \le n$.

We say that **A** is consistent if

$$a_{ij}.a_{jk} = a_{ik}, \tag{2}$$

for each $1 \le i, j, k \le n$.

If for some integer i, j, k, (2) does not hold, than **A** is said to be inconsistent. In the popular Analytic hierarchy process (AHP), it is assumed that $\frac{1}{9} \le a_{ij} \le 9$, for all $1 \le i, j \le n$, see [6]. The consistency of **A** is measured by the consistency index C*I* as

$$CI = \frac{\lambda_{max} - n}{n-1}, \tag{3}$$

where λ_{max} is the maximal eigenvalue of **A**. It can be shown that $CI \ge 0$, see [6].

If **A** is an n×n reciprocal pair-wise comparison matrix, then **A** is consistent iff $CI = 0$. To provide a measure independent of the order of the matrix, n, Saaty proposed the consistency ratio (CR). This is obtained by taking the ratio between λ_{max} - n to its expected value over a large number of positive reciprocal matrices of order n, whose entries are randomly chosen in the set of values $\left\{\frac{1}{9},...,9\right\}$. For this consistency measure, he proposed a 10% threshold for the CR to accept the estimation. In practical decision situations inconsistency is "acceptable" if CR < 0,1. For the priorization procedure based on geometric mean, the geometric consistency ratio *G*CR was proposed in [1], with an interpretation analogous to that considered for Saaty´s CR.

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3 Pair-wise comparison matrix with triangular fuzzy elements

Since using fuzzy numbers as elements of a pair-wise matrix is more expressive than using crisp values or intervals, we hope that the fuzzy approach allows a more accurate description of the decision making process. Rather than forcing the DM to provide precise representations of imprecise perceptions, we suggest using an imprecise representation instead.

We define a new consistency measure for an n×n reciprocal pair-wise comparison matrix with triangular fuzzy numbers being the elements of the matrix. The membership function of a triangular fuzzy number is piece-wise linear. There exist at least three reasons for using triangular shape of fuzzy elements:

1. The membership function of triangular fuzzy elements is piece-wise linear, i.e. relatively simple.
2. Triangular fuzzy numbers can be easily manipulated, e.g. added, multiplied etc.
3. Crisp (non-fuzzy) numbers, i.e. the most practical values, can be represented as triangular ones.

A triangular fuzzy number a can be equivalently expressed by a tripple of real numbers, i.e. $a = (a^L; a^M; a^U)$, where a^L is the Lower number, a^M is the Middle number, and a^U is the Upper number, $a^L \le a^M \le a^U$. If $a^L = a^M = a^U$, then a is said to be the crisp number (non-fuzzy number). Evidently, the set of all crisp numbers is isomorphic to the set of real numbers. In order to distinguish fuzzy and non-fuzzy numbers we shall denote fuzzy numbers, vectors and matrices by the tilde above the symbol, e.g. $\tilde{a} = (a^L; a^M; a^U)$. It is well known that the arithmetic operations +,-, * and / can be extended to fuzzy numbers by the Extension principle.

If all elements of an m×n matrix **A** are triangular fuzzy numbers we call **A** the matrix with triangular fuzzy elements and this matrix is composed of triples as follows

$$\tilde{\mathbf{A}} = \begin{bmatrix} (a_{11}^L; a_{11}^M; a_{11}^U) & \cdots & (a_{1n}^L; a_{1n}^M; a_{1n}^U) \\ \vdots & \ddots & \vdots \\ (a_{m1}^L; a_{m1}^M; a_{m1}^U) & \cdots & (a_{mn}^L; a_{mn}^M; a_{mn}^U) \end{bmatrix}.$$

Particularly, let $\tilde{\mathbf{A}}$ be an n×n matrix with triangular fuzzy elements. We say that $\tilde{\mathbf{A}}$ is reciprocal, if the following condition is satisfied: $\tilde{a}_{ij} = (a_{ij}^L; a_{ij}^M; a_{ij}^U)$ implies $\tilde{a}_{ji} = (\frac{1}{a_{ij}^U}; \frac{1}{a_{ij}^M}; \frac{1}{a_{ij}^L})$ for all i,j = 1,2,...,n, i.e.

$$\tilde{\mathbf{A}} = \begin{bmatrix} (1;1;1) & (a_{12}^L; a_{12}^M; a_{12}^U) & \cdots & (a_{1n}^L; a_{1n}^M; a_{1n}^U) \\ (\frac{1}{a_{12}^U}; \frac{1}{a_{12}^M}; \frac{1}{a_{12}^L}) & (1;1;1) & \cdots & (a_{2n}^L; a_{2n}^M; a_{2n}^U) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{a_{1n}^U}; \frac{1}{a_{1n}^M}; \frac{1}{a_{1n}^L}) & (\frac{1}{a_{2n}^U}; \frac{1}{a_{2n}^M}; \frac{1}{a_{2n}^L}) & \cdots & (1;1;1) \end{bmatrix}, \quad (4)$$

where $1 \le a_{ij}^L \le a_{ij}^M \le a_{ij}^M$, i,j=1,2,...,n. Without loss of generality we assume that $1 \le a_{ij}^M \le a_{ik}^M$ whenever i ≤ j ≤ k.

In some sense, our new inconsistency measure will become an extension of the consistency index CR. For this purpose let us consider n elements (e.g. some criteria or variants) being evaluated by the DM. The result of the pair-wise comparison evaluations of the DM is a reciprocal matrix with triangular fuzzy elements $\tilde{\mathbf{A}} = \{\tilde{a}_{ij}\}$, i.e. (4), where

$$\tilde{a}_{ji} = (\frac{1}{a_{ij}^U}; \frac{1}{a_{ij}^M}; \frac{1}{a_{ij}^L}) \quad \text{for all i},j = 1,2,...,n.$$

Here, $\frac{1}{\sigma} \le a_{ij}^L \le a_{ij}^M \le a_{ij}^U \le \sigma$ and S = $[\frac{1}{\sigma}, \sigma]$, $\sigma > 1$, is an interval of real numbers called the scale. DMs use the scale for evaluating pairs of elements. In classical AHP, $\sigma = 9$, see [6].

Now, consider deviations

$$\frac{\tilde{w}_i}{\tilde{w}_j} \simeq \tilde{a}_{ij}, \quad (5)$$

where $\tilde{w}_k = (w_k^L; w_k^M; w_k^U)$ are triangular fuzzy weights associated with evaluated elements and extended arithmetic operations on fuzzy numbers, see [2]. The question is how these weights should be calculated so that the deviations are minimized in some (fuzzy) sense.

Here, we propose a new method for calculating w_k^L, w_k^M, w_k^U by solving the following optimization problem:

$$\sum_{i,j} \left(\log \frac{w_i^L}{w_j^L} - \log a_{ij}^L \right)^2 + \left(\log \frac{w_i^M}{w_j^M} - \log a_{ij}^M \right)^2 + \left(\log \frac{w_i^U}{w_j^U} - \log a_{ij}^U \right)^2 \longrightarrow \min; \quad (6)$$

subject to

$$w_k^U \ge w_k^M \ge w_k^L \ge 0, \ k = 1, 2,...,n. \quad (7)$$

By the well known method of making the derivatives zero (a necessary condition for optimality) it can be easily shown, that the solution of problem (3), (4) shall satisfy the following relations

$$w_k^L = C_L \cdot \left(\prod_{j=1}^{n} a_{kj}^L \right)^{1/n}, w_k^M = C_M \cdot \left(\prod_{j=1}^{n} a_{kj}^M \right)^{1/n}, w_k^U = C_U \cdot \left(\prod_{j=1}^{n} a_{kj}^U \right)^{1/n} \quad (8)$$

for each k = 1, 2,...,n, where coefficients C_L, C_M, C_U are arbitrary positive constants. We shall find the values of the coefficients with the “minimal measure of fuzziness” of the associated weights.

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Moreover, we require that the middle values w_k^M of the fuzzy weights $\tilde{w}_k = (w_k^L; w_k^M; w_k^U)$ satisfy the normalization condition, i.e. $\sum_{k=1}^{n} w_k^M = 1$. Hence, from normalization condition and from (7) we obtain

$$C_M = \frac{1}{\sum_{i=1}^{n} (\prod_j a_{ij}^M)^{1/n}}. \quad (9)$$

Hence, from (6), (7) we obtain

$$C_L \le C_M \min_{i=1,\dots,n} \left\{ \frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^L\right)^{1/n}} \right\},$$

$$C_U \ge C_M \max_{i=1,\dots,n} \left\{ \frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^U\right)^{1/n}} \right\}. \quad (10)$$

Moreover, we want to find weights $\tilde{w}_k = (w_k^L; w_k^M; w_k^U)$, k = 1, 2,...,n, with the minimal spread, i.e. the minimal measure of fuzziness $s_k = w_k^U - w_k^L$. Hence, by (8), (9) we obtain the weights as follows:

$$w_k^L = C_{min} \cdot \frac{\left(\prod_{j=1}^{n} a_{kj}^L\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}, C_{min} = \min_{i=1,\dots,n} \left\{ \frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^L\right)^{1/n}} \right\}, \quad (11)$$

$$w_k^M = \frac{\left(\prod_{j=1}^{n} a_{kj}^M\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}, \quad (12)$$

$$w_k^U = C_{max} \cdot \frac{\left(\prod_{j=1}^{n} a_{kj}^U\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}, C_{max} = \max_{i=1,\dots,n} \left\{ \frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^U\right)^{1/n}} \right\}. \quad (13)$$

Particularly, if $\tilde{\mathbf{A}}$ is a crisp (i.e. nonfuzzy) matrix, i.e. $a_{ij}^L = a_{ij}^M = a_{ij}^U$ for all i, j, then $C_{min} = C_{max} = 1$, therefore, $w_k^L = w_k^M = w_k^U$ for all k. Consequently, the solution - weights are crisp, too.

4 New consistency index

Supposing the maximum of all deviations (5) for all i,j is equal to "zero", then intuitively $\tilde{\mathbf{A}}$ should be consistent. Hence, a consistency measure shall be introduced as the maximal deviation relative to the measure which could be achieved by a matrix of the same dimension.

For a given scale $S = [\frac{1}{\sigma}, \sigma]$ we define a consistency index $I_n^\sigma(\tilde{\mathbf{A}})$ of n×n reciprocal matrix $\tilde{\mathbf{A}}$ with fuzzy triangular elements as follows

$$I_n^\sigma(\tilde{\mathbf{A}}) = C_n^\sigma . \max_{i,j} \left\{ \max\{ |\frac{w_i^L}{w_j^U} - a_{ij}^L|, |\frac{w_i^M}{w_j^M} - a_{ij}^M|, |\frac{w_i^U}{w_j^L} - a_{ij}^U| \} \right\} \quad (14)$$

where w_k^L, w_k^M, w_k^U are given by (11) – (13) for all k=1,2,…,n, and

$$C_n^\sigma = \frac{1}{\max\left\{ \sigma - \sigma^{\frac{2-2n}{n}}, \sigma^2\left(\left(\frac{2}{n}\right)^{\frac{2}{n-2}} - \left(\frac{2}{n}\right)^{\frac{n}{n-2}}\right)\right\}}, \text{ if } \sigma < \left(\frac{n}{2}\right)^{\frac{n}{n-2}},$$

$$C_n^\sigma = \frac{1}{\max\left\{ \sigma - \sigma^{\frac{2-2n}{n}}, \sigma^{\frac{2n-2}{n}} - \sigma \right\}}, \text{ if } \sigma \ge \left(\frac{n}{2}\right)^{\frac{n}{n-2}}. \quad (15)$$

We say that $\tilde{\mathbf{A}}$ is F-consistent if $I_n^\sigma(\tilde{\mathbf{A}}) = 0$, otherwise, if $I_n^\sigma(\tilde{\mathbf{A}}) > 0$, then $\tilde{\mathbf{A}}$ is F-inconsistent.

The following theorem shows, that C_n^σ in (15) is a "normalizing" constant. The proof is easy, see [5].

Theorem 1. *If* $\tilde{\mathbf{A}}$ is an n×n reciprocal matrix with triangular fuzzy elements evaluated from the scale $[\frac{1}{\sigma}, \sigma]$, then

$$0 \le I_n^\sigma(\tilde{\mathbf{A}}) \le 1. \quad (16)$$

The following theorem states that the new concept of F-consistency is compatible with the old concept of consistency (2).

Theorem 2. Let $\tilde{\mathbf{A}} = \mathbf{A}$ be a non-fuzzy reciprocal pair-wise comparison matrix. *If* **A** is consistent then **A** is F-consistent.

5 Examples

In this section we present two examples showing that the newly defined consistency index is a convenient tool not only for measuring inconsistency of pair-wise comparison matrices with fuzzy elements, but also for measuring inconsistency of crisp pair-wise comparison matrices.

The first example gives the (crisp) pair-wise comparison matrix $\mathbf{A}^*$ with inconsistency index $I_n^\sigma(\mathbf{A}^*) = 1$.

Example 1. Consider 7×7 reciprocal matrix with crisp elements from the scale $[\frac{1}{9}, 9]$:

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$$\mathbf{A}^* = \begin{bmatrix} 1 & 9 & 9 & 9 & 9 & 9 & 9 \\ \frac{1}{9} & 1 & 9 & 9 & 9 & 9 & 9 \\ \frac{1}{9} & \frac{1}{9} & 1 & 9 & 9 & 9 & 9 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 1 & 9 & 9 & 9 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 1 & 9 & 9 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 1 & 9 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & 1 \end{bmatrix}.$$

Here, as $9 > \left(\frac{7}{2}\right)^{\frac{7}{7-2}} = 5.78$, we obtain

$$\frac{w_1}{w_n} - a_{1n} = \frac{\left(\sigma^{n-1}\right)^{\frac{1}{n}}}{\left(\frac{1}{\sigma^{n-1}}\right)^{\frac{1}{n}}} - \sigma = \sigma^{\frac{2n-2}{n}} - \sigma = 9^{\frac{12}{7}} - 9 = 34.24.$$

$$a_{n1} - \frac{w_n}{w_1} = \sigma - \frac{\left(\frac{1}{\sigma^{n-1}}\right)^{\frac{1}{n}}}{\left(\sigma^{n-1}\right)^{\frac{1}{n}}} = \sigma - \sigma^{\frac{2-2n}{n}} = 9 - 9^{-\frac{12}{7}} = 8.98.$$

Hence,

$$\max_{i,j}\left\{\left|\frac{w_i}{w_j} - a_{ij}\right|\right\} = \max\{34.24, 8.98\} = 34.24.$$

By (14) and (15) we get $I_7^9(\mathbf{A}^*) = 1$.

Example 2. Consider 3×3 reciprocal matrix $\widetilde{\mathbf{A}}$ with triangular fuzzy elements

$$\widetilde{\mathbf{A}} = \begin{bmatrix} (1 & 1 & 1) & (2 & 3 & 4) & (4 & 5 & 6) \\ (\frac{1}{4} & \frac{1}{3} & \frac{1}{3}) & (1 & 1 & 1) & (3 & 4 & 5) \\ (\frac{1}{6} & \frac{1}{5} & \frac{1}{4}) & (\frac{1}{5} & \frac{1}{4} & \frac{1}{3}) & (1 & 1 & 1) \end{bmatrix},$$

S = [$^1/_9$, 9] is an evaluation scale.

As $9 > \left(\frac{3}{2}\right)^{\frac{3}{3-2}} = 3.375$, we calculate

$$C_3^9 = \frac{1}{\max\left\{9 - 9^{-\frac{4}{3}}, 9^{\frac{4}{3}} - 9\right\}} = \frac{1}{\max\{8.95, 9.72\}} = 0.103.$$

By (11) – (13), we obtain $C_{max} = 0.855$, $C_{min} = 1.145$ and

$\widetilde{w}_1 = (w_1^L; w_1^M; w_1^U)$ = (0.458 ; 0.627 ; 0.627),

$\widetilde{w}_2 = (w_2^L; w_2^M; w_2^U)$ = (0.264 ; 0.280 ; 0.295),

$\widetilde{w}_3 = (w_3^L; w_3^M; w_3^U)$ = (0.094 ; 0.094 ; 0.094).

Finally, we calculate the inconsistency index $I_3^9(\widetilde{\mathbf{A}}) = 0.219$, hence $\widetilde{\mathbf{A}}$ is F-inconsistent.

For comparison with the well known AHP method, consider the corresponding crisp matrix $\mathbf{A}^M$ taking into account only the middle values of $\widetilde{\mathbf{A}}$, i.e.

$$\mathbf{A}^M = \begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{4} & 1 \end{bmatrix}.$$

Consistency index $I_3^9(\mathbf{A}^M)$= 0.174 , hence $\mathbf{A}^M$ is also F-inconsistent, the consistency index is, however, smaller than the previous one, as there is no fuzziness in $\mathbf{A}^M$. Applying consistency index by T. Saaty to matrix $\mathbf{A}^M$, we obtain CR = 0.082.

6 Conclusions

In this paper, we investigated consistency of pair-wise comparison matrix with triangular fuzzy elements based on a newly designed method of logarithmic least squares for eliciting associated weights. A new consistency index of reciprocal matrix with fuzzy triangular elements was defined. The weights we derived have the minimal measure of fuzziness among all weights based on the solution of a specific optimization problem. Some properties of the consistency index were derived and two illustrating examples were presented. The newly defined inconsistency index seems to be a convenient tool not only for measuring inconsistency of pair-wise comparison matrices with fuzzy elements, but also for measuring inconsistency of crisp pair-wise comparison matrices.

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