Universal Integrals Based on Level Dependent Capacities

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Abstract— *Three different types of universal integral based on level dependent capacities are introduced and discussed. Two extremal types are based on Caratheodory's idea of inner and outer measures, while the third type is introduced for copula-based universal integrals only.*

Keywords— Choquet integral, Copula, Fuzzy measure, Level dependent capacities, Sugeno integral, Universal integral.

1 Introduction

Capacities (also called fuzzy measures) express the weights of (measurable) subsets of a given universe X in a consistent way. If, for example, X represents a set of criteria then a capacity m on X assigns to each group of criteria $A \subseteq X$ a weight $m(A)$. Universal integrals aggregate the information contained in a capacity m and in a (measurable) function f to a single representative value. The range of f has no influence on the capacity we are exploiting in the aggregation process. In practical applications, however, there is often a need for a different treatment of functions with small values and functions with large values (or even in a more sophisticated way).

This idea of different weights for sets (groups of criteria) at different levels can be expressed by means of a system of capacities (level dependent capacities, see [1, 2, 3]). Note that, motivated by multi-criteria decision problems, one approach to a Choquet integral based on level dependent capacities was proposed and discussed in [1, 2]. On the other hand, one type of a Sugeno integral based on level dependent capacities was introduced in [3] as a solution of the comonotone maxitivite problem for aggregation functions.

The aim of this contribution is to introduce and to discuss universal integrals based on level dependent capacities, i.e., we are looking for extensions of the concept of universal integrals [4] which originally was defined for capacities only. Similar ideas generalizing classical measures are related to the notion of Markov kernels [5], see also [6].

2 Preliminaries

Recall that a *(binary) aggregation operator* or *aggregation function* [7, 8] is a function $A\colon [0,1]^2 \to [0,1]$ which is non-decreasing (in each component) and satisfies $A(0,0) = 0$ and $A(1,1) = 1$. If a binary aggregation function A has neutral element 1, i.e., satisfies $A(a,1) = A(1,a) = a$ for all $a \in [0,1]$, it is called a *semicopula* [9].

Universal integrals were introduced and studied recently in [10, 4]. If the set $\mathcal{H}$ is defined by

$$\mathcal{H} = \{h\colon [0,1] \to [0,1] \mid h \text{ is non-increasing with } h(0) = 1\}.$$

and if $\otimes\colon [0,1]^2 \to [0,1]$ is a semicopula then a non-decreasing functional $J\colon \mathcal{H} \to [0,1]$ is called $\otimes$-*consistent* if, for all $a, b \in [0,1]$, we have

$$J(b \cdot \mathbf{1}_{[0,a]} + (1-b) \cdot \mathbf{1}_{\{0\}}) = a \otimes b. \tag{1}$$

A *capacity space* $(X, \mathcal{A}, m)$ is a triplet consisting of a non-empty universe X, a σ-algebra $\mathcal{A}$ of subsets of X (the elements of $\mathcal{A}$ are called *measurable subsets of* X), and a capacity $m\colon \mathcal{A} \to [0,1]$, i.e., m is isotone with boundary conditions $m(\emptyset) = 0$ and $m(X) = 1$. Furthermore, denote by $\mathcal{F}$ the system of all $\mathcal{A}$-measurable functions $f\colon X \to [0,1]$. Note that, in our context, a function $f\colon X \to [0,1]$ is called $\mathcal{A}$-*measurable* if, for each Borel subset B of $[0,1]$, its preimage $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ is a measurable subset of X, i.e., belongs to the σ-algebra $\mathcal{A}$. For details about capacity spaces (also called *fuzzy measure spaces*) see [11, 12, 13, 14].

Definition 2.1 Let $J\colon \mathcal{H} \to [0,1]$ be a $\otimes$-consistent functional. A J-*universal integral* $\mathbf{I}_J$ is a mapping which can be defined for each capacity space $(X, \mathcal{A}, m)$ via $\mathbf{I}_{J,m}\colon \mathcal{F} \to [0,1]$ given by

$$\mathbf{I}_{J,m}(f) = J(h_{m,f}), \tag{2}$$

where the function $h_{m,f}\colon [0,1] \to [0,1]$ is defined by $h_{m,f}(t) = m(\{f \geq t\})$ (observe that, because of the monotonicity of m, we have $h_{m,f} \in \mathcal{H}$).

Note that two prototypical universal integrals are the *Choquet integral* [15, 16, 17], in which case we have $J = R$, the standard Riemann integral given by

$$R(h) = \int_0^1 h(t)\, dt \tag{3}$$

(here $\otimes = \Pi$, the standard product), and the *Sugeno integral* [18, 11, 12, 17] which is related to $J = S$ with

$$S(h) = \sup\{\min(t, h(t)) \mid t \in [0,1]\} \tag{4}$$

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(here $\otimes = \text{Min}$, the greatest semicopula).

Next, we recall two important classes of universal integrals (for more details see [10, 4]):

Proposition 2.2 *Let $\otimes$ be a semicopula. The smallest universal integral $\mathbf{I}_\otimes$ based on $\otimes$ is given by $J_\otimes \colon \mathcal{H} \to [0,1]$,*

$$J_\otimes(h) = \sup\{t \otimes h(t) \mid t \in [0,1]\}, \tag{5}$$

i.e., $\mathbf{I}_\otimes = \mathbf{I}_{J_\otimes}$.

Evidently, because of (5), we have

$$\mathbf{I}_{\otimes,m}(f) = \sup\{t \otimes m(\{f \geq t\}) \mid t \in [0,1]\}. \tag{6}$$

Observe that for $\otimes = \text{Min}$, $\mathbf{I}_{\text{Min}} = \mathbf{I}_S$ is just the Sugeno integral, while $\mathbf{I}_\Pi$ is the *Shilkret integral* [19]. If $T \colon [0,1]^2 \to [0,1]$ is a strict t-norm [20], then $\mathbf{I}_T$ is the *Weber integral* [21].

Recall that a *(two-dimensional) copula* [22] is a binary aggregation function $C \colon [0,1]^2 \to [0,1]$ with annihilator 0 and neutral element 1, i.e., satisfies $C(a,0) = C(0,a) = 0$ and $C(a,1) = C(1,a) = a$ for all $a \in [0,1]$, which is also 2-*increasing*, i.e., for all $a_1, a_2, b_1, b_2 \in [0,1]$ with $a_1 \leq a_2$ and $b_1 \leq b_2$ we have

$$C(a_1,b_1) - C(a_1,b_2) + C(a_2,b_2) - C(a_2,b_1) \geq 0. \tag{7}$$

This means that each copula C induces a probability measure P_C on the Borel subsets of $[0,1]^2$ via

$$\begin{aligned} &P_C([a_1,a_2] \times [b_1,b_2]) \\ &\quad = C(a_1,b_1) - C(a_1,b_2) + C(a_2,b_2) - C(a_2,b_1). \end{aligned} \tag{8}$$

Equivalently, a copula C is a semicopula which is *supermodular*, i.e.,

$$\begin{aligned} &C(a_1,b_1) + C(a_2,b_2) \\ &\quad = C((a_1,b_1) \wedge (a_2,b_2)) + C((a_1,b_1) \vee (a_2,b_2)) \end{aligned} \tag{9}$$

for all $(a_1,b_1),(a_2,b_2) \in [0,1]^2$, where $\wedge$ and $\vee$ are the (pointwise) lattice operations on $[0,1]^2$, i.e, $\wedge = \min$ and $\vee = \max$.

Proposition 2.3 *If $C \colon [0,1]^2 \to [0,1]$ is a copula and P_C the probability measure on the Borel subsets of $[0,1]^2$ induced by C, then the functional $J_C \colon \mathcal{H} \to [0,1]$ given by*

$$J_C(h) = P_C(\{(x,y) \in [0,1]^2 \mid y \leq h(x)\}) \tag{10}$$

is C-consistent.

Given a copula $C \colon [0,1]^2 \to [0,1]$, we shall denote the universal integral $\mathbf{I}_{J_C}$ simply by $\mathbf{I}_{(C)}$. Since Π and Min are copulas, we see that $\mathbf{I}_{(\Pi)} = \mathbf{I}_R$ is the Choquet integral and $\mathbf{I}_{(\text{Min})} = \mathbf{I}_S$ is the Sugeno integral. Therefore, we have two different ways to define the Sugeno integral.

3 Level dependent capacities

The notion of level dependent capacities was introduced in [1], see also [2].

Definition 3.1 Let $(X,\mathcal{A})$ be a measurable space. A *level dependent capacity* on $(X,\mathcal{A})$ is a system $M = (m_t)_{t\in[0,1]}$, where each $m_t \colon \mathcal{A} \to [0,1]$ is a capacity on $(X,\mathcal{A})$.

A special example of a level dependent capacity is a *Markov kernel* [5, 6], where each m_t is a probability measure on $(X,\mathcal{A})$ and, for each $A \in \mathcal{A}$, the function $M_A \colon [0,1] \to [0,1]$ given by $M_A(t) = m_t(A)$ is $\mathcal{A}$-measurable.

Given a level dependent capacity $M = (m_t)_{t\in[0,1]}$ on $(X,\mathcal{A})$, for each $\mathcal{A}$-measurable function $f \colon X \to [0,1]$ we define the function $h_{M,f} \colon [0,1] \to [0,1]$, which accumulates all the information contained in M and f, by

$$h_{M,f}(t) = m_t(\{f \geq t\}). \tag{11}$$

In general, the function $h_{M,f}$ is neither monotone nor even $\mathcal{A}$-measurable (compare with the function $h_{m,f} \in \mathcal{H}$ discussed in Section 2). Following the ideas of inner and outer measures in Caratheodory's approach [23], we introduce the two functions $(h_{M,f})^*, (h_{M,f})_* \colon [0,1] \to [0,1]$ by

$$(h_{M,f})^* = \inf\{h \in \mathcal{H} \mid h \geq h_{M,f}\}, \tag{12}$$
$$(h_{M,f})_* = \sup\{h \in \mathcal{H} \mid h \leq h_{M,f}\}. \tag{13}$$

Note that it is possible to show that for all $t \in [0,1]$

$$(h_{M,f})^*(t) = \sup\{h_{M,f}(u) \mid u \in [t,1]\}, \tag{14}$$
$$(h_{M,f})_*(t) = \inf\{h_{M,f}(u) \mid u \in [0,t]\}. \tag{15}$$

Evidently, both functions $(h_{M,f})^*$ and$(h_{M,f})_*$ are non-increasing and, therefore, belong to $\mathcal{H}$. If the level dependent capacity $M = (m_t)_{t\in[0,1]}$ is constant (i.e., $m_t = m$ for all $t \in [0,1]$) then the three functions considered in (11)–(13) coincide, i.e., we have

$$h_{m,f} = h_{M,f} = (h_{M,f})^* = (h_{M,f})_*. \tag{16}$$

The three functions $h_{M,f}$, $(h_{M,f})^*$ and $(h_{M,f})_*$ allow us to introduce three different extensions of universal integrals for level dependent capacities.

4 Extensions of universal integrals

If $\mathbf{I}_J$ is a J-universal integral then a mapping $\widetilde{\mathbf{I}}_J$ which can be defined on arbitrary measurable spaces $(X,\mathcal{A})$ and arbitrary level dependent capacities M on $(X,\mathcal{A})$ is called an *extension of* $\mathbf{I}_J$ if, whenever the level dependent capacity $M = (m_t)_{t\in[0,1]}$ is constant (i.e., $m_t = m$ for all $t \in [0,1]$), we have $\widetilde{\mathbf{I}}_{J,M} = \mathbf{I}_{J,m}$.

Using the functions in (12) and (13), two extremal extensions of universal integrals can be introduced.

Definition 4.1 Let $\mathbf{I}_J$ be a J-universal integral. The *upper extension* $(\mathbf{I}_J)^*$ (respectively the *lower extension* $(\mathbf{I}_J)_*$) of $\mathbf{I}_J$ for an arbitrary measurable space $(X,\mathcal{A})$, a level dependent capacity M on $(X,\mathcal{A})$, and an $\mathcal{A}$-measurable function $f \in \mathcal{F}$ are given by, respectively,

$$(\mathbf{I}_{J,M})^*(f) = J((h_{M,f})^*), \tag{17}$$
$$(\mathbf{I}_{J,M})_*(f) = J((h_{M,f})_*). \tag{18}$$

If the system $M = (m_t)_{t\in[0,1]}$ is constant (i.e., $m_t = m$ for all $t \in [0,1]$) then we clearly have

$$\mathbf{I}_{J,m} = (\mathbf{I}_{J,M})_* = (\mathbf{I}_{J,M})^*, \tag{19}$$

i.e., both $(\mathbf{I}_J)_*$ and $(\mathbf{I}_J)^*$ are indeed extensions of $\mathbf{I}_J$.

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For extensions $\widetilde{\mathbf{I}}_{J,M}$ of $\mathbf{I}_J$ which are non-decreasing in M we get

$$(\mathbf{I}_{J,M})_* \leq \widetilde{\mathbf{I}}_{J,M} \leq (\mathbf{I}_{J,M})^*. \tag{20}$$

The approach of extending a known integral to more general situations was applied, e.g., in the case of belief and plausibility measures. Indeed, in the case of a belief measure m [14, 16], the standard Lebesque integral (for probability measures) was applied to probability measures $P \geq m$, and the integral with respect to the capacity m (which, in general, is non-additive) was in this case defined as the infimum of all Lebesgue integrals (with respect to probability measures P with $P \geq m$). Similarly, for a plausibility measure m, all probability measures $P \leq m$ were taken into account, and then the supremum over all Lebesgue integrals (with respect to probability measures P with $P \leq m$) yields the integral for the (non-additive) plausibility measure m. Observe that in both cases the resulting integral is the Choquet integral with respect to m. In our case, having a universal integral $\mathbf{I}_J$ defined for any pair (m, f) of a capacity and a measurable function on the same space, we can compare such pairs based on the corresponding $h_{m,f}$: $(m_1, f_1) \leq (m_2, f_2)$ whenever $h_{m_1,f_1} \leq h_{m_2,f_2}$ (here (m_1, f_1) and (m_2, f_2) need not be defined on the same measurable space, in general). Then $(\mathbf{I}_{J,M})^*(f)$ is just the infimum of all values $(\mathbf{I}_{J,m})(g)$, the infimum being taken over all (m, g) with $h_{M,f} \leq h_{m,g}$. Similarly, $(\mathbf{I}_{J,M})_*(f)$ is just the supremum of $(\mathbf{I}_{J,m})(g)$, where the supremum is taken over all (m, g) with $h_{M,f} \geq h_{m,g}$.

For a copula-based universal integral $\mathbf{I}_C$ there is a third extension — however, it cannot necessarily be applied to any measurable function $f \in \mathcal{F}$.

Definition 4.2 Let C be a copula, $(X, \mathcal{A})$ be a measurable space and $M = (m_t)_{t\in[0,1]}$ be a level dependent capacity on $(X, \mathcal{A})$.

(i) A function $f \in \mathcal{F}$ is called M-*integrable* if $h_{M,f}$ is a measurable function.

(ii) For each M-integrable function $f \in \mathcal{F}$ the corresponding C-based universal integral $\widetilde{\mathbf{I}}_{(C)}$ is defined by $\widetilde{\mathbf{I}}_{(C),M}(f) = J_C(h_{M,f})$ (compare (10)), i.e.,

$$\widetilde{\mathbf{I}}_{(C),M}(f) = P_C(\{(u, v) \in [0, 1]^2 \mid v \leq h_{M,f}(u)\}). \tag{21}$$

Note that a similar extension is possible in the case of a universal integral $\mathbf{I}_\mu$ based on a capacity μ on the Borel subsets of $]0, 1[^2$ as proposed in [24], in which case

$$\mathbf{I}_{\mu,M}(f) = \mu(\{(u, v) \in]0, 1[^2 \mid v < h_{M,f}(u)\}). \tag{22}$$

Remark 4.3

(i) The generalization of the Choquet integral for level dependent capacities as proposed in [1] is closely related to the Riemann-integrability of the function $h_{M,f}$. For example, if X is a finite set (and $\mathcal{A} = 2^X$), and if the level dependent capacity $M = (m_t)_{t\in[0,1]}$ has the same measurability property as a Markov kernel, i.e., for each $A \subseteq X$ the function $M_A : [0, 1] \to [0, 1]$ given by $M_A(t) = m_t(A)$ is measurable, then also $h_{M,f}$ is measurable for each $f : X \to [0, 1]$. Since $h_{M,f}$ is not necessarily monotone, the Riemann integral in the original definition of the Choquet integral should be replaced by the Lebesgue integral (with respect to the standard Lebesgue measure λ on the Borel subsets of $[0, 1]$), i.e., then (21) turns into

$$\widetilde{\mathbf{I}}_{(\Pi),M}(f) = \int_{[0,1]} h_{M,f}\, d\lambda. \tag{23}$$

However, based on Definition 4.1, we have two other extensions of the Choquet integral given by

$$(\mathbf{I}_{(\Pi),M})^*(f) = \int_0^1 (h_{M,f})^*(x)\, dx, \tag{24}$$

$$(\mathbf{I}_{(\Pi),M})_*(f) = \int_0^1 (h_{M,f})_*(x)\, dx. \tag{25}$$

(ii) Similarly, we have three possible extensions of the Sugeno integral, namely,

$$\widetilde{\mathbf{I}}_{(\mathrm{Min}),M}(f) = \lambda(\{t \in [0, 1] \mid t \leq h_{M,f}(t)\}), \tag{26}$$

$$(\mathbf{I}_{(\mathrm{Min}),M})^*(f) = \sup_{t\in[0,1]} \min(t, h_{M,f}(t)), \tag{27}$$

$$(\mathbf{I}_{(\mathrm{Min}),M})_*(f) = \sup_{t\in[0,1]} \min\Big(t, \inf_{u\in[0,t]} h_{M,f}(u)\Big). \tag{28}$$

In [3] comonotone maxitivity of aggregation functions was investigated and, without any reference to integrals, $(\mathbf{I}_{(\mathrm{Min}),M})^*$ was found to be a solution, compare also [25].

Example 4.4 Let $X = [0, 1]$, $\mathcal{A}$ be the σ-algebra of Borel subsets of $[0, 1]$, and define $M = (m_t)_{t\in[0,1]}$ by

$$m_t = \begin{cases} m^* & \text{if } t \in \left[0, \frac{1}{4}\right], \\ \sqrt{\lambda} & \text{if } t \in \left[\frac{1}{2}, 1\right], \\ m_* & \text{otherwise,} \end{cases} \tag{29}$$

where m^* and m_* are the greatest and the smallest capacity on $(X, \mathcal{A})$, respectively, given by

$$m^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{otherwise,} \end{cases} \tag{30}$$

$$m_*(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

If $f = \mathrm{id}_{[0,1]}$ then we get

$$(h_{M,f})^* = \mathbf{1}_{[0,\frac{1}{4}]} + \tfrac{1}{\sqrt{2}} \cdot \mathbf{1}_{]\frac{1}{4},\frac{1}{2}[} + \sqrt{1-f} \cdot \mathbf{1}_{[\frac{1}{2},1]}, \tag{32}$$

$$h_{M,f} = \mathbf{1}_{[0,\frac{1}{4}]} + \sqrt{1-f} \cdot \mathbf{1}_{[\frac{1}{2},1]}, \tag{33}$$

$$(h_{M,f})_* = \mathbf{1}_{[0,\frac{1}{4}]}. \tag{34}$$

Consequently, we have for the corresponding extensions of the Choquet integral

$$(\mathbf{I}_{(\Pi),M})^*(f) \approx 0.663, \tag{35}$$

$$\widetilde{\mathbf{I}}_{(\Pi),M}(f) \approx 0.486, \tag{36}$$

$$(\mathbf{I}_{(\Pi),M})_*(f) = 0.25. \tag{37}$$

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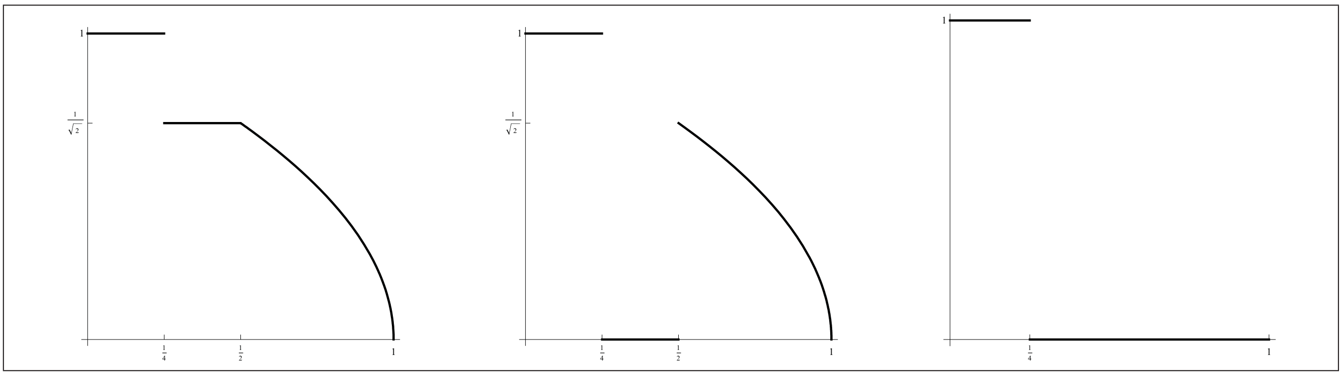

Figure 1: The functions $(h_{M,f})^*$ (left), $h_{M,f}$ (center), and $(h_{M,f})_*$ in Example 4.4

For the extensions of the Sugeno integral we get

$$(\mathbf{I}_{(\mathrm{Min}),M})^*(f) \approx 0.618, \quad (38)$$
$$\widetilde{\mathbf{I}}_{(\mathrm{Min}),M}(f) \approx 0.368, \quad (39)$$
$$(\mathbf{I}_{(\mathrm{Min}),M})_*(f) = 0.25. \quad (40)$$

The Shilkret integral [19] can be considered either as the smallest universal integral based on $\otimes = \Pi$ or as μ-based universal integral, where μ is the capacity on the Borel subsets of $]0,1[^2$ given by $\mu(E) = \sup\{x \cdot y \mid (x,y) \in E\}$. Then

$$(\mathbf{I}_{(\Pi),M})^*(f) \approx 0.385, \quad (41)$$
$$\widetilde{\mathbf{I}}_{\mu,M}(f) \approx 0.385, \quad (42)$$
$$(\mathbf{I}_{(\Pi),M})_*(f) = 0.25. \quad (43)$$

5 Conclusions

Universal integrals based on level dependent capacities can be seen as natural extensions of capacity-based universal integrals acting on different subdomains with (possibly) different capacities. Take, for example, $X = \{1,2\}$ and define the capacities $v_1, v_2 \colon 2^X \to [0,1]$ by $v_1(\{1\}) = \frac{1}{3}$, $v_1(\{2\}) = \frac{2}{3}$, and $v_2(\{1\}) = \frac{3}{4}$, $v_2(\{2\}) = \frac{1}{4}$. Both capacities are additive (i.e., discrete probability measures), and the corresponding Choquet integrals are just weighted arithmetic means, i.e., $W_1(x,y) = \frac{x+2y}{3}$ and $W_2(x,y) = \frac{3x+y}{4}$. Consider the level dependent capacity $M = (m_t)_{t\in[0,1]}$ given by

$$m_t = \begin{cases} v_1 & \text{if } t \le \frac{1}{2}, \\ v_2 & \text{otherwise.} \end{cases} \quad (44)$$

Then, for $(x,y) \in \left[0, \frac{1}{2}\right]^2$, we have

$$(\mathbf{I}_{(\Pi),M})^*(x,y) = \widetilde{\mathbf{I}}_{(\Pi),M}(x,y) = (\mathbf{I}_{(\Pi),M})_*(x,y) = W_1(x,y), \quad (45)$$

and for $(x,y) \in \left[\frac{1}{2}, 1\right]^2$

$$(\mathbf{I}_{(\Pi),M})^*(x,y) = \widetilde{\mathbf{I}}_{(\Pi),M}(x,y) = (\mathbf{I}_{(\Pi),M})_*(x,y) = W_2(x,y), \quad (46)$$

If $(x,y) \in \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$ then

$$(\mathbf{I}_{(\Pi),M})^*(x,y) = \widetilde{\mathbf{I}}_{(\Pi),M}(x,y) = (\mathbf{I}_{(\Pi),M})_*(x,y) = \tfrac{8x+6y+5}{24}. \quad (47)$$

However, our three extensions of the Choquet integral lead to three different extensions of W_1 (restricted to $\left[0, \frac{1}{2}\right]^2$) and W_2 (restricted to $\left[\frac{1}{2}, 1\right]^2$): if $(x,y) \in \left]\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right[$ then

$$\widetilde{\mathbf{I}}_{(\Pi),M}(x,y) = \tfrac{18x+16y-5}{24}, \quad (48)$$
$$(\mathbf{I}_{(\Pi),M})^*(x,y) = W_2(x,y), \quad (49)$$
$$(\mathbf{I}_{(\Pi),M})_*(x,y) = W_1(x,y). \quad (50)$$

Note that $\widetilde{\mathbf{I}}_{(\Pi),M}$ is a continuous aggregation function while $(\mathbf{I}_{(\Pi),M})^*$ and $(\mathbf{I}_{(\Pi),M})_*$ are non-continuous. Finally, observe that $\widetilde{\mathbf{I}}_{(\Pi),M}$ is the ordinal sum extension of the aggregation functions W_1 and W_2 as proposed in [26].

We expect applications of the functionals introduced here in multi-criteria decision making, especially in situations when the weights of the criteria are related to the cardinal values of the score values.

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References

[1] S. Giove, S. Greco, and B. Matarazzo. Level dependent capacities and the generalized Choquet integral. In D. Dubois, E. P. Klement, and R. Mesiar, editors, *Proceedings 28th Linz Seminar on Fuzzy Set Theory: "Fuzzy Sets, Probability, and Statistics—Gaps and Bridges"*, pages 54–55, Linz (Austria), 2007. Johannes Kepler Universität.

[2] S. Greco, S. Giove, and B. Matarazzo. The Choquet integral with respect to a level dependent capacity. *Fuzzy Sets and Systems*. (in press).

[3] R. Mesiar, A. Mesiarová-Zemánková, and K. Ahmad. Level-dependent Sugeno integral. *IEEE Trans. Fuzzy Systems*. (in press).

[4] E. P. Klement, R. Mesiar, and E. Pap. A universal integral as common frame for Choquet and Sugeno integral. Submitted for publication.

[5] H. Bauer. *Probability Theory and Elements of Measure Theory*. Academic Press, London, 1981.

[6] E. P. Klement. Characterization of finite fuzzy measures using Markoff-kernels. *J. Math. Anal. Appl.*, 75:330–339, 1980.

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[7] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar. Aggregation operators: properties, classes and construction methods. In Calvo et al. [8], pages 3–104.

[8] T. Calvo, G. Mayor, and R. Mesiar, editors. *Aggregation Operators. New Trends and Applications*, volume 97 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2002.

[9] F. Durante and C. Sempi. Semicopulæ. *Kybernetika (Prague)*, 41:315–328, 2005.

[10] E. P. Klement, R. Mesiar, and E. Pap. A universal integral. In *Proceedings EUSFLAT 2007, Ostrava*, volume I, pages 253–256. 2007.

[11] Z. Wang and G. J. Klir. *Fuzzy Measure Theory*. Plenum Press, New York, 1992.

[12] E. Pap. *Null-Additive Set Functions*. Kluwer Academic Publishers, Dordrecht, 1995.

[13] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy Measures and Integrals. Theory and Applications*, volume 40 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2000.

[14] E. Pap, editor. *Handbook of Measure Theory*. Elsevier Science, Amsterdam, 2002.

[15] G. Choquet. Theory of capacities. *Ann. Inst. Fourier (Grenoble)*, 5:131–292, 1953–1954.

[16] D. Denneberg. *Non-additive Measure and Integral*. Kluwer Academic Publishers, Dordrecht, 1994.

[17] P. Benvenuti, R. Mesiar, and D. Vivona. Monotone set functions-based integrals. In Pap [14], chapter 33, pages 1329–1379.

[18] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

[19] N. Shilkret. Maxitive measure and integration. *Indag. Math.*, 33:109–116, 1971.

[20] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic. Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.

[21] S. Weber. Two integrals and some modified versions: critical remarks. *Fuzzy Sets and Systems*, 20:97–105, 1986.

[22] R. B. Nelsen. *An Introduction to Copulas*, volume 139 of *Lecture Notes in Statistics*. Springer, New York, second edition, 2006.

[23] P. R. Halmos. *Measure Theory*. Van Nostrand Reinhold, New York, 1950.

[24] E. P. Klement, R. Mesiar, and E. Pap. Measure-based aggregation operators. *Fuzzy Sets and Systems*, 142:3–14, 2004.

[25] M. Couceiro and J.-L. Marichal. Representations and characterizations of polynomial functions on chains. arXiv:0811.0309v, 2008.

[26] B. De Baets and R. Mesiar. Ordinal sums of aggregation operators. In B. Bouchon-Meunier, J. Gutiérrez-Ríos, L. Magdalena, and R. R. Yager, editors, *Technologies for Constructing Intelligent Systems*, volume 2: Tools, pages 137–148. Physica-Verlag, Heidelberg, 2002.

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