Games and capacities on partitions

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Abstract— We consider capacities and games whose value $v(S)$ depend on the organization of $N \setminus S$, this organization being represented by a partition. Hence we deal with capacities and games depending on two arguments: a subset (coalition) S and a partition π containing S. We call embedded coalition any such pair (S, π). We first provide a suitable structure for the set of embedded coalitions, study its properties, and find the Möbius function for this structure. Then, we define games and capacities on this structure and study their properties. We propose also a concept similar to the Shapley value, and provide an axiomatization of it. Lastly, we give some insights on the Choquet integral.

Keywords— capacity, fuzzy measure, partition, Möbius transform, Shapley value

1 Introduction

Capacities [1] or fuzzy measures [2] are usually defined on the set of subsets of the universe, assumed to be finite in this paper, or on a subcollection of it. Specifically, denoting by N the universe, let us take a capacity v defined on 2^N, and consider some subset $S \in 2^N$ together with the quantity $v(S)$. The interpretation of $v(S)$ may differ according to the context, but usually falls into two categories: either $v(S)$ represents a degree of certainty, plausibility, etc. that the true (but unknown) state of the world is contained in S, or $v(S)$ represents some power, importance, strength, monetary value of the group S of entities (agents, players, voters, criteria, etc.). Using a common word, we may say that $v(S)$ is the *worth* of S.

Let us consider more carefully the second interpretation, which is related to cooperative game theory. If N represents a society of individuals, speaking of $v(S)$ implicitly means that the society is formed of the group (or coalition) S and the coalition $N \setminus S$. Under this organization, the worth of S is $v(S)$. We may however consider less simple situations, where the set of remaining agents $N \setminus S$ is also organized in some way, say $N \setminus S = S_2 \cup S_3 \cup \cdots \cup S_k$, where $\{S_2, S_3, \ldots, S_k\}$ is a partition of $N \setminus S$. Then it is natural to think that the worth of S may depend on the organization of the remaining agents. Therefore, the expression $v(S)$ is no more enough precise, and we need to consider $v(S, \{S, S_2, S_3, \ldots, S_k\})$, for all possible organizations of the society containing S as a block of the society. The usual term for the argument of v, namely (S, π) with π the partition $\{S, S_2, S_3, \ldots, S_k\}$, is *embedded coalition*, and v is called a *game in partition function form*, or *PFF-game* for short [3]. We will call them in this paper games (or capacities) on partitions.

Although games on partitions have been studied in the game theoretic literature (essntially the Shapley value, the core), no explicit study has been done on the mathematical object (S, π) and its structure, nor on the properties of v. The aim of this paper is precisely to fill this gap. We will provide a structure for embedded coalitions, study its properties, and find the Möbius function on this structure. Then we will study properties of games, and define a Shapley value for v, and finally give some remarks on the possibility to define a Choquet-like integral. A part of this paper is based on the working paper [4].

2 The structure of embedded coalitions

Let $N := \{1, 2, \ldots, n\}$ be the universal set (set of agents, players, etc.). We denote by $S, T, \ldots$ the subsets of N (coalitions), and their cardinality by $s, t, \ldots$. We consider the set $\Pi(N)$ (denoted also by $\Pi(n)$) of all possible partitions of N (coverings of N by disjoint subsets). For a partition $\pi := \{S_1, \ldots, S_k\}$, subsets $S_1, \ldots, S_k$ are called *blocks* of π. A partition in k blocks is called a *k-partition*. A natural ordering of partitions is given by the notion of "coarsening" or "refinement". Taking π, π' partitions in $\Pi(N)$, we say that π is a *refinement* of π' (or π' is a *coarsening* of π), denoted by $\pi \leq \pi'$, if any block of π is contained in a block of π' (or every block of π' fully decomposes into blocks of π). Then $(\Pi(N), \leq)$ is a lattice, called the *partition lattice*. With this ordering, the bottom element of the lattice is the finest partition $\pi^\perp := \{\{1\}, \ldots, \{n\}\}$, while the top element is the coarsest partition $\pi^\top := \{N\}$.

An *embedded coalition* is a pair (S, π), where $S \in 2^N \setminus \{\emptyset\}$, and $\pi \in \Pi(N)$ is such that $S \in \pi$. We denote by $\mathfrak{C}(N)$ (or by $\mathfrak{C}(n)$) the set of embedded coalitions on N. For the sake of concision, we often denote by $S\pi$ the embedded coalition (S, π), and omit braces and commas for subsets (example with $n = 3$: $12\{12, 3\}$ instead of $(\{1,2\}, \{\{1,2\}, \{3\}\})$). Remark that $\mathfrak{C}(N)$ is a proper subset of $2^N \times \Pi(N)$.

As mentionned in the introduction, works on games on partitions do not explicitly define a structure (that is, some order) on embedded coalitions. A natural choice is to take the product order on $2^N \times \Pi(N)$:

$$(S, \pi) \sqsubseteq (S', \pi') \Leftrightarrow S \subseteq S' \text{ and } \pi \leq \pi'.$$

Evidently, the top element of this ordered set is $(N, \pi^\top)$ (denoted more simply by $N\{N\}$ according to our conventions). However, due to the fact that the empty set is not allowed in (S, π), there is no bottom element in the ordered structure $(\mathfrak{C}(N), \sqsubseteq)$. Indeed, all elements of the form $(\{i\}, \pi^\perp)$ are minimal elements. For mathematical convenience, we introduce an artificial bottom element $\perp$ to $\mathfrak{C}(N)$ (it could be considered as $(\emptyset, \pi^\perp)$), and denote $\mathfrak{C}(N)_\perp := \mathfrak{C}(N) \cup \{\perp\}$. We give as illustration the partially ordered set $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ with $n = 3$ (Fig. 1).

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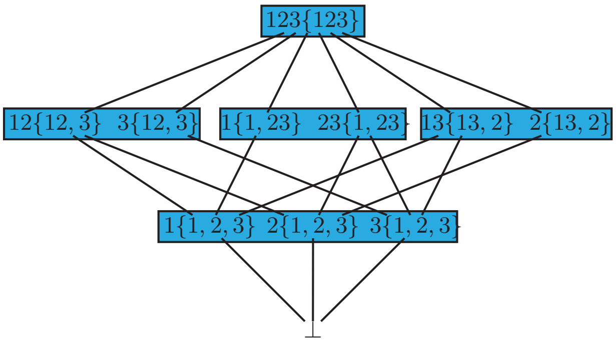

Figure 1: Hasse diagram of $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ with $n = 3$. Elements with the same partition are framed in grey.

Another possibility to define an order on embedded coalitions is to basically take the Boolean lattice 2^N, and for any $S \in 2^N$, duplicate it into $S\pi_1, S\pi_2, \ldots$ for all partitions π containing S. Then $(S, \pi) \leq (S', \pi')$ if and only if $S \subseteq S'$. This seems to be the underlying structure in the work of Macho-Stadler et al. [5]. We illustrate this structure for $n = 3$ in Figure 2.

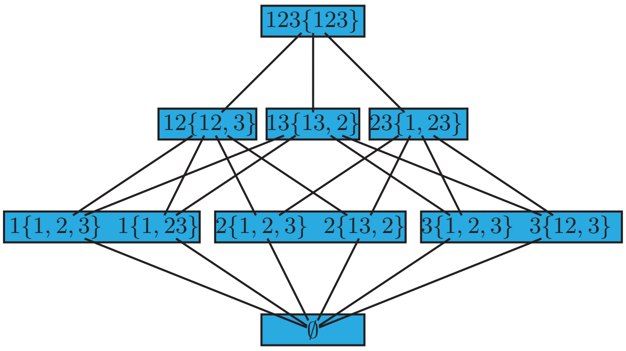

Figure 2: Hasse diagram of the structure of Macho-Stadler et al. for $n = 3$. Elements with same coalition (duplicates) are framed in grey.

In the sequel, we focus on the first structure, since it has better properties. The second one is not a lattice, since for example, $12\{12, 3\}$ and $13\{13, 2\}$ are two minimal upper bounds of $1\{1, 2, 3\}$ and $1\{1, 23\}$.

We recall that in a partially ordered set $(P, \leq)$ with a bottom element $\perp$ and a top element $\top$, a *chain from* $\perp$ *to* $\top$ is a totally ordered sequence of elements of P including $\perp, \top$. The chain is *maximal* if no other chain can contain it (equivalently, if between two consecutive elements x_i, x_{i+1} of the sequence, there is no element $x \in P$ such that $x_i < x < x_{i+1}$). If no ambiguity occurs, we say *maximal chain* instead of *maximal chain from* $\perp$ *to* $\top$. The set of maximal chains in P is denoted by $\mathcal{C}(P)$. The *length* of a maximal chain is the number of elements of the sequence minus 1.

The following is shown in [4].

Proposition 1. The following holds.

(i) $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is an upper semimodular lattice, whose top and bottom elements are $(N, \pi^\top)$ and $\perp$. All elements are complemented: for a given $S\pi$, any embedded coalition of the form $\overline{S}\pi_{\overline{S}}$ with $\overline{S}$ the complement of S and $\pi_{\overline{S}}$ any partition containing $\overline{S}$.

(ii) Every maximal chain from $\perp$ to an element (S, π) has the same length, which is $n - k + 1$, if π is a partition in k blocks. Hence, the height of the lattice is n.

(iii) The total number of elements of $\mathfrak{C}(N)_\perp$ is $\sum_{k=1}^{n} kS_{n,k} + 1$, where $S_{n,k}$ is the Stirling number of second kind.

n	1	2	3	4	5	6	7	8
$\lvert\mathfrak{C}(n)_\perp\rvert$	2	4	11	38	152	675	3264	17008

(iv) The number of maximal chains from $\perp$ to $N\{N\}$ is $\frac{(n!)^2}{2^{n-1}}$, which is also the number of maximal longest chains in $\mathfrak{C}(N)$.

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(\mathfrak{C}(n)_\perp)\rvert$	1	2	9	72	900	16 200	396 900	12 700 800

(v) Let (S, π) be an embedded coalition, with $\pi := \{S, S_2 \ldots, S_k\}$, and $|S| = s$. The number of maximal chains from $\perp$ to (S, π) is

$$|\mathcal{C}([\perp, (S, \pi)])| = \frac{s(n-k)!}{2^{n-k}} s! s_2! \cdots s_k!.$$

For example, there are 9 chains from $\perp$ to $123\{123, 4\}$.

(vi) Let (S, π) be an embedded coalition, with $\pi := \{S, S_2 \ldots, S_k\}$. The number of maximal chains from (S, π) to $N\{N\}$ is

$$|\mathcal{C}([(S, \pi), N\{N\}])| = \frac{1}{k} |\mathcal{C}(\mathfrak{C}(k)_\perp)| = \frac{k!(k-1)!}{2^{k-1}}.$$

The above results focus on maximal chains, since they are a fundamental concept for the sequel, especially for the Shapley value. It is important to note that this lattice is neither distributive, modular nor atomic (and hence not geometric). Atoms are all elements of the form $\{i\}\pi^\perp$, and elements representable by a join of atoms are only those of the form $S\pi_S^\perp$, where $\pi_S^\perp = \{S, i_1, \ldots, i_{n-s}\}$, with $N \setminus S = \{i_1, \ldots, i_{n-s}\}$.

3 Games and capacities on partitions

Definition 1. A *game on partitions* on N is a mapping $v : \mathfrak{C}(N)_\perp \to \mathbb{R}$, such that $v(\perp) = 0$. The set of all games on partitions on N is denoted by $\mathcal{PG}(N)$.

Although any such function could be called a game on partitions, to be meaningful, we need to assume that forming the grand coalition generates the largest total surplus, i.e., $v(N\{N\}) \geq \sum_{S \in \pi} v(S, \pi)$, for all $\pi \in \Pi(N)$.

The structure of lattice permits to define the usual notions on games and capacities.

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Definition 2. Let $v \in \mathcal{PG}(N)$.

(i) v is *monotone* if $S\pi \sqsubseteq S'\pi'$ implies $v(S\pi) \leq v(S'\pi')$. A monotone game on partitions is called a *capacity on partitions*. A capacity on partitions v is normalized if $v(N\{N\}) = 1$.

(ii) v is *supermodular* if for every $S\pi, S'\pi'$ we have

$$v(S\pi \vee S'\pi') + v(S\pi \wedge S'\pi') \geq v(S\pi) + v(S'\pi').$$

It is *submodular* if the reverse inequality holds.

(iii) A game is *additive* if it is both supermodular and submodular.

(iv) More generally, for a given $k \geq 2$, a game is *k-monotone* if for all families of k elements $S_1\pi_1, \ldots, S_k\pi_k$, we have

$$v(\bigvee_{i\in K} S_i\pi_i) \geq \sum_{J\subseteq K, J\neq\emptyset} (-1)^{|J+1|} v(\bigwedge_{i\in J} S_i\pi_i)$$

putting $K := \{1, \ldots, k\}$. A game is ∞-monotone if it is k-monotone for every $k \geq 2$.

(v) A game is a belief function if it is a normalized ∞-monotone capacity.

Above, $\vee, \wedge$ are the least upper bounds and greatest lower bounds. The greatest lower bound is simply given by:

$$(S, \pi) \wedge (S', \pi') = (S \cap S', \pi \wedge \pi') \text{ if } S \cap S' \neq \emptyset,$$

and $\perp$ otherwise. The least upper bound is more tricky to define:

$$(S, \pi) \vee (S', \pi') = (T \cup T', \rho)$$

where T, T' are blocks of $\pi \vee \pi'$ containing respectively S and S', and ρ is the partition obtained by merging T and T' in $\pi \vee \pi'$.

The following result is due to Barthélemy [6].

Proposition 2. Let L be a lattice. Then f is monotone and ∞-monotone on L if and only if it is monotone and $(|L| - 2)$-monotone.

In lattice theory, a *valuation* (or *2-valuation*) on a lattice L is a real-valued function on L being both super- and submodular (additive). More generally, for a given $k \geq 2$, a *k-valuation* satisfies

$$v(\bigvee_{i\in K} x_i) = \sum_{J\subseteq K, J\neq\emptyset} (-1)^{|J+1|} v(\bigwedge_{i\in J} x_i)$$

for every family of k elements. An ∞-valuation is a function f which is a k-valuation for every $k \geq 2$. The following results are useful (see [7, Ch. X], and also [6]).

Proposition 3. Let L be a lattice.

(i) L is modular if and only if it admits a strictly monotone valuation v (i.e., $x < y$ implies $v(x) < v(y)$).

(ii) L is distributive if and only if it admits a strictly monotone 3-valuation.

(iii) L is distributive if and only if it is modular and every strictly monotone valuation is a k-valuation for any $k \geq 2$.

(iv) Any lattice admits an ∞-valuation.

The consequence of (i) is that no strictly monotone additive game exists since $\mathfrak{C}(n)_\perp$ is not modular when $n > 2$. The question is: does it exist an additive game? The following proposition answers this question.

Proposition 4. $\mathfrak{C}(2)_\perp$ admits a strictly monotone 2-valuation (hence by Prop. 2, a strictly monotone ∞-monotone valuation). If $n > 2$, the only possible valuations on $\mathfrak{C}(n)_\perp$ are constant valuations.

Since any game v satisfies $v(\perp) = 0$, we have:

Corollary 1. The only additive game is the constant game $v = 0$.

Definition 3. Let $T\sigma \in \mathfrak{C}(N)$.

(i) The *unanimity game centered on* $T\sigma$ is the game defined by

$$u_{T\sigma}(S\pi) = \begin{cases} 1, & \text{if } S\pi \sqsupseteq T\sigma \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The *identity game centered on* $T\sigma$ is the game defined by

$$\delta_{T\sigma}(S\pi) = \begin{cases} 1, & \text{if } S\pi = T\sigma \\ 0, & \text{otherwise.} \end{cases}$$

Unanimity games are normalized capacities while the latter ones are not. Identity games form a basis of the vector space $\mathcal{PG}(N)$ since for any $v \in \mathcal{PG}(N)$,

$$v = \sum_{T\sigma\in\mathfrak{C}(N)} v(T\sigma)\delta_{T\sigma}.$$

From Proposition 1 (iii), the dimension of the basis is $\sum_{k=1}^{n} kS_{n,k}$. A less trivial basis is given by unanimity games, through the Möbius transform, described in the next paragraph.

In capacity theory and partially ordered sets, a key point is to determine the Möbius function, enabling to compute the Möbius transform of games and capacities. We recall that for a partially ordered set $(P, \leq)$ which is locally finite, the Möbius function $\mu(x, y) : P^2 \to \mathbb{R}$ permits to solve the equation:

$$g(x) = \sum_{y\leq x} f(y)$$

where $f, g : P \to \mathbb{R}$, as follows:

$$f(x) = \sum_{y\leq x} g(y)\mu(y, x).$$

(see Rota [8], Aigner [9]). Based on properties of $\mathfrak{C}(N)_\perp$, the following fundamental result:

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Proposition 5. If P is a lattice with bottom element 0 and set of atoms $\mathcal{A}$, for every $x \in P$, the Möbius function reads

$$\mu(0, x) = \sum_{S \subseteq \mathcal{A} | \bigvee S = x} (-1)^{|S|},$$

and the fact that for the partition lattice we know that $\mu_{\Pi(n)}(\pi^\perp, \{N\}) = (-1)^{n-1}(n-1)!$ (see Aigner [9, p. 154]), the following can be shown.

Proposition 6. The Möbius function on $\mathfrak{C}(n)_\perp$ is given by:

$$\mu(\perp, S\pi) = \begin{cases} (-1)^{|S|}, & \text{if } \pi = \pi_S^\perp \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

$$\mu(S'\pi', S\pi) = (-1)^{k'-k}(l_1 - 1)! \cdots (l_k - 1)!, \quad (2)$$

for every $S'\pi' \leq S\pi$, where $\pi_S^\perp$ is the partition consisting in S and all singletons in $N \setminus S$, $S\pi = S\{S, S_2, \ldots, S_k\}$, $S'\pi' = S'\{S', S_{12}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{k1}, \ldots, S_{kl_k}\}$, $S_i = S_{i1} \cup \cdots \cup S_{il_i}$ for $i = 2, \ldots, k$, and $k' := \sum_{i=1}^k l_i$.

Using this, the Möbius transform of any PFF-game v is given by

$$m(S\pi) = \sum_{S'\pi' \sqsubseteq S\pi} \mu(S'\pi', S\pi) v(S'\pi').$$

Now, $m(S\pi)$ gives the coordinates of v in the basis of unanimity games.

Example 1. Application for $n = 3$. Let us compute the Möbius transform of a game v. We have, for all distinct $i, j, k \in \{1, 2, 3\}$:

$$\begin{aligned} m(\perp) &= 0 \\ m(i\pi^\perp) &= v(i\pi^\perp) \\ m(ij\{ij, k\}) &= v(ij\{ij, k\}) - v(i\pi^\perp) - v(j\pi^\perp) \\ m(i\{i, jk\}) &= v(i\{i, jk\}) - v(i\pi^\perp) \\ m(123\{123\}) &= v(123\{123\}) - \sum_{i,j} v(ij\{ij, k\}) \\ &\quad - \sum_i v(i\{i, jk\}) + 2\sum_i v(i\pi^\perp). \end{aligned}$$

Barthélemy [6] proved the following.

Proposition 7. Let L be any lattice, and $f : L \to \mathbb{R}$. If the Möbius transform of f, denoted by m, satisfies $m(\perp) = 0$, $m(x) \geq 0$ for all $x \in L$, and $\sum_{x \in L} m(x) = 1$ (normalization), then f is a belief function.

The converse of this proposition does not hold in general (it holds for the Boolean lattice 2^N). A belief function is *invertible* if its Möbius transform is nonnegative, normalized and vanishes at $\perp$. The following counterexample shows that for $\mathfrak{C}(n)_\perp$, there exist belief functions which are not invertible.

Example 2. Let us take $\mathfrak{C}(3)_\perp$, and consider a function f whose values are given on the figure below. Monotonicity implies that $1 \geq \beta \geq \alpha \geq 0$. In order to check ∞-monotonicity,

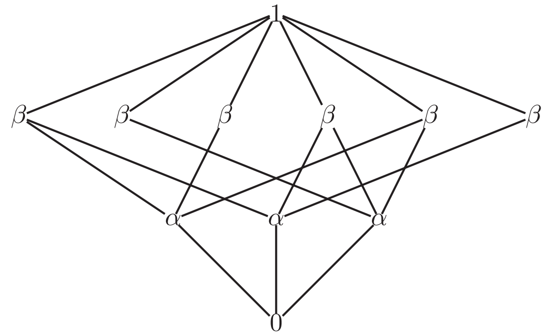

Figure 3: Hasse diagram of $(\mathfrak{C}(3)_\perp, \sqsubseteq)$

from Proposition 2 we know that it suffices to check till 7-monotonicity. We write below the most constraining inequalities only, keeping in mind that $1 \geq \beta \geq \alpha \geq 0$.

2-monotonicity is equivalent to $\beta \geq 2\alpha$ and $1 \geq 2\beta - \alpha$. 3-monotonicity is equivalent to $1 \geq 3\beta - \alpha$. 4-monotonicity is equivalent to $1 \geq 4\beta - 3\alpha$. 5-monotonicity is equivalent to $1 \geq 5\beta - 4\alpha$, while 6-monotonicity is equivalent to $1 \geq 6\beta - 9\alpha$. 7-monotonicity does not add further constraints.

From Example 1, nonnegativity of the Möbius transform implies $\alpha \geq 0$ (atoms), $\beta \geq 2\alpha$ (2nd level), and $6\alpha - 6\beta + 1 \geq 0$ for the top element. Then taking $\alpha = 0.1$, $\beta = 0.28$ make that f is a belief function, but $m(\top) = -0.08$.

The last point concerns necessity functions. A *necessity function* on a lattice L is a real-valued function f on L such that

$$f(x \wedge y) = \min(f(x), f(y)).$$

The following proposition by Barthélemy shows that necessity functions always exist on $\mathfrak{C}(n)_\perp$.

Proposition 8. Let L be a lattice and $f : L \to \mathbb{R}$ such that $f(\perp) = 0$, $f(\top) = 1$. Then f is a necessity function if and only if it is an invertible belief function whose Möbius transform is nonzero on a chain of $\mathfrak{C}(n)_\perp$.

In other words, taking any chain C in $\mathfrak{C}(n)_\perp$ and assigning nonnegative numbers (at least one of them being positive) on elements of C such that their sum is 1 generates (by Proposition 7) a belief function which is a necessity function.

4 The Shapley value

A key notion in game theory and also in applications of capacity theory is the Shapley value [10]. For classical games, it defines a rational way to share the quantity $v(N)$ among the individuals in N. Our aim is to define a similar notion for games on partitions. We give a simplified exposition of [4]. See also Macho-Stadler et al. [5] for another proposition based on their structure.

Basically, our Shapley value is defined through maximal chains in $\mathfrak{C}(N)_\perp$, which are called *scenarios*, because they depict a particular story of coalition formation, starting from the society of individuals and arriving at the grand coalition. The set of scenarios is denoted by $\mathfrak{S}$.

In a scenario $\mathcal{S}$, some elements play a special role. We consider those elements $S\pi$ such that in the sequence of elements of $\mathcal{S}$ from bottom to top, $S\pi$ is the last element with S. They are called *terminal elements*. The set of terminal elements in $\mathcal{S}$ is denoted by $\mathcal{F}(\mathcal{S})$. For example, in the following scenario with $N = \{1, 2, 3, 4\}$, terminal elements are in bold:

$$\mathcal{S}_1 = \mathbf{1\{1, 2, 3, 4\}} \to 13\{13, 2, 4\} \to \mathbf{13\{13, 24\}} \to \mathbf{N\{N\}}.$$

Basically, a value represents the average contribution of players in the game. The contribution of player i is defined as the sum of his contribution in each scenario. However, this contribution is more complex to define than in classical games. Taking as example the above scenario $\mathcal{S}_1$, we see that three situations can arise:

(i) In one step, a single player enters: this happens for steps 1 and 2. This is exactly like classical games, and the contribution goes for the entering player.

(ii) In one step, several players enter together: this happens in the last step where players 2 and 4 enter together. In this case, the principle of insufficient reason tells us to divide the contribution equally among the entering players.

(iii) In one step, no new player enters: this happens in step 3 of $\mathcal{S}_1$. In this case, we wait till a new change occurs (i.e., a new player enters). Hence these steps are not taken into account.

Note that the third case implies that only steps which correspond to terminal elements are taken into account. Applying this methodology to the above scenario $\mathcal{S}_1$, we obtain:

(i) contribution of player 1: $v(1\{1, 2, 3, 4\}) - 0$

(ii) contribution of player 2: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13, 24\}))$

(iii) contribution of player 3: $v(13\{13, 24\}) - v(1\{1, 2, 3, 4\})$

(iv) contribution of player 4: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13, 24\}))$

Based on the above considerations, we can define the contribution of a player in a scenario.

Definition 4. The *contribution of player i in a given scenario* $\mathcal{S}$ is given by

$$\Delta_i^{\mathcal{S}}(v) := \frac{1}{|S' \setminus S|}(v(S'\pi') - v(S\pi)),$$

where $S\pi$ is the last element where i is not present (note that $S\pi \in \mathcal{F}(\mathcal{S})$), and $S'\pi'$ is the next terminal element.

Based on the idea of scenario, we define a scenario-value, from which the value will be constructed.

Definition 5. (i) A *scenario-value* is a mapping $\phi : \mathcal{PG} \to \mathbb{R}^{n\times|\mathfrak{S}|}$. Components of $\phi(v)$ are denoted by $\phi_i^{\mathcal{S}}(v)$ for scenario $\mathcal{S}$ and player i.

The *Shapley scenario-value* is defined by

$$\phi_i^{\mathcal{S}}(v) := \Delta_i^{\mathcal{S}}(v), \quad i \in N, \mathcal{S} \in \mathfrak{S}.$$

(ii) A *value* is a mapping $\phi : \mathcal{PG} \to \mathbb{R}^n$. Components of $\phi(v)$ are denoted by $\phi_i(v)$ for player i. Any scenario-value ϕ induces a value by:

$$\phi_i(v) := \frac{1}{c}\sum_{\mathcal{S}\in\mathfrak{S}} \phi_i^{\mathcal{S}}(v),$$

where c is the number of maximal chains in $\mathfrak{C}(N)_\perp$. The *Shapley value of* v is induced by the Shapley scenario-value.

It is possible to give a direct expression of the Shapley value without using the scenario-value, by expressing the game in the basis of identity games and using linearity of the Shapley value. Using the generic notation $T\sigma := T\{T, T_2, \ldots, T_k\}$, we have

$$\begin{aligned}\phi_i(v) = &\frac{1}{n}v(N\{N\})+\\ &\sum_{T\sigma, T\ni i, T\neq N} \frac{2(n-k)!}{n!n!}(k-1)(k-1)!(k-2)!\times\\ &t!t_2!\cdots t_k!v(T\sigma)\\ &-\sum_{T\sigma, T\not\ni i} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!\times\\ &t!(t_2-1)!\cdots t_k!v(T\sigma),\end{aligned}$$

where it is assumed in the third term that $i \in T_2$, the second block of σ. An equivalent expression, although less computationally efficient but closer to the classical Shapley value, is:

$$\begin{aligned}\phi_i(v) = &\frac{1}{n}v(N\{N\})\\ &+\sum_{T\sigma, T\not\ni i, T_2\supset\{i\}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!\times\\ &t!(t_2-1)!\cdots t_k!\Big[\frac{t+1}{t}v(T\cup i\sigma_{T\cup i}) - v(T\sigma)\Big]\\ &-\sum_{T\sigma, T\not\ni i, T_2=\{i\}} \frac{2t(n-k)!}{n!n!}(k-1)!(k-2)!\times\\ &t!t_3!\cdots t_k!v(T\sigma),\end{aligned}$$

with $\sigma_{T\cup i}$ the partition obtained from σ by moving $i \in T_2$ to T. With $n = 3$, we obtain:

$$\begin{aligned}\phi_1(v) = &\frac{1}{3}v(N\{N\}) + \frac{1}{9}v(12\{12, 3\}) + \frac{1}{9}v(13\{13, 2\})\\ &-\frac{2}{9}v(23\{23, 1\}) + \frac{1}{9}v(1\{1, 23\}) - \frac{1}{18}v(2\{2, 13\})\\ &-\frac{1}{18}v(3\{3, 12\}) + \frac{2}{9}v(1\{1, 2, 3\}) - \frac{1}{9}v(2\{1, 2, 3\})\\ &-\frac{1}{9}v(3\{1, 2, 3\}).\end{aligned}$$

We introduce axioms for the characterization of the Shapley scenario-value.

Definition 6. (i) A value is *efficient (E)* if $\sum_{i\in N}\phi_i(v) = v(N\{N\})$.

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(ii) A scenario-value is *scenario-efficient (SE)* if $\sum_{i\in N} \phi_i^{\mathcal{S}}(v) = v(N\{N\})$ for all $\mathcal{S} \in \mathfrak{S}$.

Evidently, scenario-efficiency implies efficieny of the induced value.

Proposition 9. The Shapley scenario-value is scenario-efficient. Hence, the Shapley value is efficient as well.

A scenario-value satisfies *linearity (L)* if it is linear on $\mathcal{PG}(N)$.

Definition 7. Let us consider $i \in N$, a scenario $\mathcal{S}$, and denote by $S\pi$ the last element in $\mathcal{S}$ not containing i, and $S'\pi'$ its successor in $\mathcal{F}(\mathcal{S})$. Player i is *null in scenario* $\mathcal{S}$ *for* v if $v(S\pi) = v(S'\pi')$.

Null axiom (N): If i is null in scenario $\mathcal{S}$ for v, then $\phi_i^{\mathcal{S}}(v) = 0$.

The *symmetry axiom for the scenario-value (SS)* reads: For any $i \in N$, any $\mathcal{S} \in \mathfrak{S}$, and any permutation σ on N, it holds

$$\phi_i^{\mathcal{S}}(v) = \phi_{\sigma(i)}^{\sigma(\mathcal{S})}(v \circ \sigma^{-1})$$

with $\sigma(\mathcal{S}), \sigma(S, \pi)$ defined naturally as follows: $\sigma(S, \pi) = (\sigma(S), \sigma(\pi))$, where $\sigma(S) = \{\sigma(i) \mid i \in S\}$, $\sigma(\pi) = \{\sigma(S) \mid S \in \pi\}$, and $\sigma(\mathcal{S}) = \{\sigma(S, \pi) \mid (S, \pi) \in \mathcal{S}\}$.

We given the following simple characterization.

Proposition 10. The Shapley scenario-value is the unique scenario-value satisfying (L), (N), (SS), and (SE).

Another axiomatization avoiding the strong (SE) axiom is given in [4].

5 The Choquet integral

We end the paper by giving some considerations on the Choquet integral defined for capacities on partitions. The (classical) Choquet integral in its discrete version is also based on maximal chains. Specifically, taking a function $f : N \to \mathbb{R}_+$, the maximal chain induced by f is given by

$$\{\sigma(n)\}, \{\sigma(n), \sigma(n-1)\}, \ldots, \{\sigma(n), \ldots, \sigma(1)\}$$

where σ is any permutation such that

$$f_{\sigma(1)} \leq f_{\sigma(2)} \leq \cdots \leq f_{\sigma(n)}$$

where $f(i)$ is denoted by f_i for simplicity. Then the Choquet integral of f is defined by:

$$\int f\, dv := \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) v(\{\sigma(n), \ldots, \sigma(i)\}),$$

with the convention $f_{\sigma(0)} = 0$. Let us keep this idea for capacities on partitions, and consider for simplicity a function f such that $f_1 < f_2 < \cdots < f_n$. Then there is only one maximal chain in $\mathfrak{C}(N)_\perp$ induced by f, containing the subsets $\{n\}, \{n, n-1\}, \ldots, N$, which is

$$\{n\}\pi^\perp, \{n, n-1\}\pi^\perp_{\{n,n-1\}}, \ldots, N\{N\}.$$

Hence the Choquet integral should be defined by:

$$\int f\, dv := \sum_{i=1}^{n} (f_i - f_{i-1}) v(\{n, \ldots, i\}\{\{n, \ldots, i\}, 1, 2, \ldots, i-1\}).$$

Consider now that $f_1 = f_2 < f_3 \cdots < f_n$. Since two permutations can order this function, and the result should be the same whatever the permutation used, it is not difficult to check that this can be achieved only if the above formula is used, thus excluding the use of embedded coalitions which are not of the form $S\pi_S^\perp$. But then it suffices to put $\tilde{v}(S) := v(S\pi_S^\perp)$, and we are back to the classical Choquet integral.

As a conclusion, it does not seem to make sense to consider capacities on partitions for the Choquet integral.

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