The Minimization of the Risk of Falling in Portfolios under Uncertainty

Yuji Yoshida
Faculty of Economics and Business Administration, University of Kitakyushu
4-2-1 Kitagata, Kokuraminami, Kitakyushu 802-8577, Japan
Email: yoshida@kitakyu-u.ac.jp

Abstract— *A portfolio model to minimize the risk of falling under uncertainty is discussed. The risk of falling is represented by the value-at-risk of rate of return. Introducing the perception-based extension of the value-at-risk, this paper formulates a portfolio problem to minimize the risk of falling with fuzzy random variables. In the proposed model, randomness and fuzziness are evaluated respectively by the probabilistic expectation and the mean with evaluation weights and λ-mean functions. The analytical solutions of the portfolio problem regarding the risk of falling are derived. This paper gives formulae to show the explicit relations among the following important parameters in portfolio: The expected rate of return, the risk probability of falling and bankruptcy, and the rate of falling regarding the asset prices. A numerical example is given to explain how to obtain the optimal portfolio and these parameters from the asset prices in the stock market. Several figures are shown to observe the relation among these parameters at the optimal portfolios.*

Keywords— Value-at-risk (VaR), risk-sensitive portfolio, fuzzy random variable, perception-based extension, the risk probability, the rate of falling.

1 Introduction

In financial market, the portfolio is one of the most useful risk allocation technique for stable asset management. The minimization of the financial risk as well as the maximization of the return are important themes in the asset management. In a classical portfolio theory, *Markowitz's mean-variance model* is studied by many researchers and fruitful results have been achieved, and then the variance is investigated as the risk for portfolios ([8, 9, 11]). In this paper we focus on the drastic decline of asset prices. Recently, *value-at-risk (VaR)* is used widely to estimate the risk that asset prices decline based on worst scenarios. VaR is a risk-sensitive criterion based on percentiles, and it is one of the standard criteria in asset management ([17]). VaR is a kind of risk-level values of the asset at a specified probability of decline and it is used to select portfolios after due consideration of worst scenarios in investment. Many researchers and financial traders usually use VaR by mathematical programming since it is not easy to analyze the VaR portfolio model mathematically. Because Markowitz's mean-variance criterion and the variance-minimizing criterion are represented by quadratic programming, but VaR criterion in portfolio is neither linear nor quadratic ([17]). In this paper, by use of VaR regarding the rates of return, we discuss a *portfolio selection problem not only to minimize the rates of falling and but also to maximize the expected rates of return*. In the proposed portfolio model, we can maximize the expected rate of return after due consideration of the worst scenarios. This paper derives analytical solutions for the portfolio problem to minimize the rate of falling. The main purpose in this paper is to derive the explicit relations among the important parameters γ, p and δ regarding the financial risk from the obtained analytical results, where γ is the expected rate of return, p is the risk probability of falling and bankruptcy, and δ is the rate of falling regarding the asset at the optimal portfolios.

Soft computing like fuzzy logic works well for financial models in uncertain environment. Estimation of uncertain quantities is important in decision making. To represent uncertainty in this portfolio model, we use *fuzzy random variables* which have two kinds of uncertainties, i.e. randomness and fuzziness. In this paper, randomness is used to represent the uncertainty regarding the belief degree of frequency, and fuzziness is applied to linguistic imprecision of data because of a lack of knowledge regarding the current stock market. At the financial crisis in October 2008, we have observed the serious distrust of the market that the risky information regarding banks and security companies, for example the amounts of trouble loans, risky accounts, debts and so on, may not disclose to the investors and the public, and it is surely a kind of risks occurring from the imprecision of information. The fuzziness comes from the imprecision of data because of a lack of knowledge, and such serious distrust in the stock market will be represented by the fuzziness of information in finance models. We extend the VaR for real random variables to one regarding fuzzy random variables from the viewpoint of perception-based approach in Yoshida [14]. We formulate the a portfolio problem with fuzzy random variables, and we discuss the fundamental properties of the extended VaR. Recently, Yoshida [13] introduced the mean, the variance and the measurement of fuzziness of fuzzy random variables, using *evaluation weights* and λ-mean functions. This paper estimates fuzzy numbers and fuzzy random variables by the probabilistic expectation and these criteria, which are characterized by *possibility and necessity criteria* for subjective estimation and a *pessimistic-optimistic index* for subjective decision. These parameters are decided by the investor and are based on the degree of his certainty regarding the current information in the market. In this portfolio model, we use triangle-type fuzzy numbers and fuzzy random variables for computation in actual models. Regarding falling and bankruptcy, we observe the direct relations among the rate of falling, the expected rate of return and the risk probability at the optimal portfolios by figures derived from the obtained analytical results.

2 A portfolio model and the rate of falling

In this section, we explain a portfolio model with n stocks, where n is a positive integer. Let $\mathbb{T} := \{0, 1, 2, \cdots, T\}$ be the time space with an expiration date T, and $\mathbb{R}$ denotes the set of all real numbers. Let (Ω, P) be a probability space, where P is a non-atomic probability on a sample space Ω. For an asset $i = 1, 2, \cdots, n$, a *stock price process* $\{S_t^i\}_{t=0}^T$ is given

ISBN: 978-989-95079-6-8

by *rates of return* R_t^i as follows. Let

$$S_t^i := S_{t-1}^i(1 + R_t^i) \tag{1}$$

for $t = 1, 2, \cdots, T$, where $\{R_t^i\}_{t=1}^T$ is assumed to be an integrable sequence of independent real random variables. Hence $w_t = (w_t^1, w_t^2, \cdots, w_t^n)$ is called a *portfolio weight vector* if it satisfies $w_t^1 + w_t^2 + \cdots + w_t^n = 1$, and further a portfolio $(w_t^1, w_t^2, \cdots, w_t^n)$ is said to *allow for short selling* if $w_t^i \geq 0$ for all $i = 1, 2, \cdots, n$. Then the rate of return with a portfolio $(w_t^1, w_t^2, \cdots, w_t^n)$ is given by

$$R_t := w_t^1 R_t^1 + w_t^2 R_t^2 + \cdots + w_t^n R_t^n. \tag{2}$$

Therefore, the reward at time $t = 1, 2, \cdots, T$ follows

$$S_t := S_{t-1} \sum_{i=1}^n w_t^i (1 + R_t^i) = S_{t-1}(1 + R_t). \tag{3}$$

In this paper, we present a portfolio model where stock price processes S_t^i take fuzzy values using fuzzy random variables. The falling of asset prices is one of the most important risks in stock markets. In this section, we discuss a portfolio model where the risk is estimated by the rate of falling. Regarding the asset (3) with the portfolio w_t, the theoretical *bankruptcy* at time t occurs on scenarios ω satisfying $S_t(\omega) \leq 0$, i.e. it follows $1 + R_t(\omega) \leq 0$ from (3). Similarly, for a constant δ satisfying $0 \leq \delta \leq 1$, a set of sample paths

$$\{\omega \in \Omega | 1 + R_t(\omega) \leq 1 - \delta\} = \{\omega \in \Omega | R_t(\omega) \leq -\delta\} \tag{4}$$

is the event of scenarios where the asset price S_t will fall from the current price S_{t-1} to a lower level than $100(1 - \delta)\,\%$ of the current price S_{t-1}, i.e. the rate of falling is $100\,\delta\,\%$. Then we say shortly that the asset is $100\,\delta\,\%$*-falling* at time t and the parameter δ is called *the rate of falling*. The probability of $100\,\delta\,\%$-falling is also given by

$$p_\delta := P(R_t \leq -\delta). \tag{5}$$

For example, p_δ denotes the probability of the falling below par value if '$\delta = 0$' and it indicates the probability of the bankruptcy if '$\delta = 1$'. In this paper, we discuss portfolios to minimize the rate of falling δ.

For a positive probability p, a value-at-risk (VaR) regarding the rate of return R_t at the probability p is given by a real number v satisfying

$$P(R_t \leq v) = p \tag{6}$$

since P is non-atomic. The value-at-risk v is the upper bound of the rate of return R_t at the worst scenarios under a given risk probability p, and then the value-at-risk v in (6) is denoted by $\mathrm{VaR}_p(R_t)$. From (5) and (6), for a risk probability $p = p_\delta$, the rate of falling is

$$\delta = -\mathrm{VaR}_p(R_t). \tag{7}$$

To minimize the rate of falling (7) under a fuzzy and random environment, in next section we discuss the fundamental properties of value-at-risks.

3 A portfolio model with fuzzy random variables

In this section, we introduce fuzzy numbers and fuzzy random variables and we give a portfolio model under uncertainty. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous and quasi-concave and has a compact support ([18]). $\mathcal{R}$ denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with their corresponding membership functions. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\}$ $(\alpha \in (0, 1])$ and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. We write the closed intervals as $\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ for $\alpha \in [0, 1]$. Hence we also introduce a partial order $\succeq$, so called the *fuzzy max order*, on fuzzy numbers $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. Then, $\tilde{a} \succeq \tilde{b}$ means that $\tilde{a}_\alpha^- \geq \tilde{b}_\alpha^-$ and $\tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+$ for all $\alpha \in [0, 1]$. An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined by Zadeh's extension principle as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\xi \in \mathbb{R}$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\xi\tilde{a}$ of ξ and $\tilde{a}$ are fuzzy numbers given by their α-cuts $(\tilde{a}+\tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+]$, $(\tilde{a}-\tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]$ and $(\xi\tilde{a})_\alpha := [\xi\tilde{a}_\alpha^-, \xi\tilde{a}_\alpha^+]$ if $\xi \geq 0$.

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* if the maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in (0, 1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ ([7, 10]). We need to introduce expectations of fuzzy random variables in order to describe a portfolio model. A fuzzy random variable $\tilde{X}$ is said to be integrably bounded if both $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in (0, 1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number $E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}$ for $x \in \mathbb{R}$, where $E(\tilde{X})_\alpha := [\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)]$ for $\alpha \in (0, 1]$ ([6, 10]).

Now we deal with a case where the rate of return $\{R_t^i\}_{t=1}^T$ has some imprecision. In this paper, we use triangle-type fuzzy random variables for computation, however we can apply the similar approach to general fuzzy random variables. We define a *rate of return process with imprecision* $\{\tilde{R}_t^i\}_{t=0}^T$ by a sequence of triangle-type fuzzy random variables

$$\tilde{R}_t^i(\cdot)(x) = \begin{cases} 0 & \text{if } x < R_t^i - c_t^i \\ \frac{x - R_t^i + c_t^i}{c_t^i} & \text{if } R_t^i - c_t^i \leq x < R_t^i \\ \frac{x - R_t^i - c_t^i}{-c_t^i} & \text{if } R_t^i \leq x < R_t^i + c_t^i \\ 0 & \text{if } x \geq R_t^i + c_t^i, \end{cases} \tag{8}$$

where c_t^i is a positive number. We call c_t^i a *fuzzy factor* for asset i at time t. Hence we can represent $\tilde{R}_t^i$ by the sum of the real random variable R_t^i and a fuzzy number $\tilde{a}_t^i$:

$$\tilde{R}_t^i(\omega)(\cdot) := 1_{\{R_t^i(\omega)\}}(\cdot) + \tilde{a}_t^i(\cdot) \tag{9}$$

for $\omega \in \Omega$, where $1_{\{\cdot\}}$ denotes the characteristic function of a singleton and $\tilde{a}_t^i$ is a triangle-type fuzzy number defined by

$$\tilde{a}_t^i(x) = \begin{cases} 0 & \text{if } x < -c_t^i \\ \frac{x + c_t^i}{c_t^i} & \text{if } -c_t^i \leq x < 0 \\ \frac{x - c_t^i}{-c_t^i} & \text{if } 0 \leq x < c_t^i \\ 0 & \text{if } x \geq c_t^i. \end{cases} \tag{10}$$

ISBN: 978-989-95079-6-8

For assets $i = 1, 2, \cdots, n$, we define *stock price processes* $\{\tilde{S}_t^i\}_{t=0}^T$ by the *rates of return with imprecision* $\tilde{R}_t^i$ as follows: $\tilde{S}_0^i := S_0^i$ is a positive number and

$$\tilde{S}_t^i = \tilde{S}_0^i \prod_{s=1}^{t} (1 + \tilde{R}_s^i) \tag{11}$$

for $t = 1, 2, \cdots, T$. For a portfolio $w = (w^1, w^2, \cdots, w^n)$, the rate of return with imprecision is given by a linear combination of fuzzy random variables

$$\tilde{R}_t := w^1 \tilde{R}_t^1 + w^2 \tilde{R}_t^2 + \cdots + w^n \tilde{R}_t^n. \tag{12}$$

In Section 4 we investigate the value-at-risk to apply the fuzzy random variable (12), and in Section 5 we discuss the portfolio problem to minimize the rate of falling regarding (12).

4 A perception-based extension of VaR

First we introduce mathematical notations of the value-at-risk for real random variables to apply it to the rates of return (12). Let $\mathcal{X}$ be the set of all integrable real random variables X on Ω with a continuous distribution function $x \mapsto F_X(x) := P(X < x)$ for which there exists a non-empty open interval I such that $F_X(\cdot) : I \mapsto (0,1)$ is a strictly increasing and onto. Then there exists a strictly increasing and continuous inverse function $F_X^{-1} : (0,1) \mapsto I$. We note that $F_X(\cdot) : I \mapsto (0,1)$ and $F_X^{-1} : (0,1) \mapsto I$ are one-to-one and onto, and we put $F_X(\inf I) := \lim_{x \downarrow \inf I} F_X(x) = 0$ and $F_X(\sup I) := \lim_{x \uparrow \sup I} F_X(x) = 1$. Then, the *value-at-risk*, shortly for *VaR*, at a risk probability p is given by the $100\,p$-percentile of the distribution function F_X: For a probability p ($0 < p < 1$),

$$\mathrm{VaR}_p(X) := \sup\{x \in I \mid F_X(x) \le p\}, \tag{13}$$

and then we have $F_X(\mathrm{VaR}_p(X)) = p$ and $\mathrm{VaR}_p(X) = F_X^{-1}(p)$ for $0 < p < 1$. In this paper, we assume that VaR v in (6) has the following representation (14).

$$\begin{aligned}(\mathrm{VaR}\ v) \ &= \ (\text{the mean}) \ - \ (\text{a positive constant } \kappa) \\ &\qquad \times (\text{the standard deviation}),\end{aligned} \tag{14}$$

where the positive constant κ is given corresponding to the probability p. The details are as follows: For any probability p satisfying $0 < p < 1$, there exists a positive constant κ such that a real number $v := \mu_t - \kappa\,\sigma_t$ satisfies (14) for all portfolios, where μ_t and σ_t are the expectation and the standard deviation of R_t respectively. One of the most popular sufficient condition for (14) is what the distribution of the rate of return R_t is normal ([2, 17]). In this paper, we obtain an exact estimation of κ in Theorem 1.

Let $\tilde{\mathcal{X}}$ be the set of all fuzzy random variables $\tilde{X}$ on Ω such that their α-cuts $\tilde{X}_\alpha^\pm$ are integrable and $\lambda \tilde{X}_\alpha^- + (1-\lambda)\tilde{X}_\alpha^+ \in \mathcal{X}$ for all $\lambda \in [0,1]$ and $\alpha \in [0,1]$. Hence, from (8) we introduce a value-at-risk for a fuzzy random variable $\tilde{X} (\in \tilde{\mathcal{X}})$ at a positive risk probability p as follows.

$$\mathrm{VaR}_p(\tilde{X})(x) := \sup_{X \in \mathcal{X} \,:\, \mathrm{VaR}_p(X) = x} \ \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \tag{15}$$

for $x \in \mathbb{R}$. Yoshida [14] has studied *perception-based estimations* extending the concept of the expectations in Kruce and Meyer [6]. This definition (15) is an extension from the value-at-risk on real random variables to one on fuzzy random variables based on the perception. The value-at-risk (15) on fuzzy random variables is characterized by the following representation, which is from the continuity and the monotonicity of $\mathrm{VaR}_p(\cdot)$.

Lemma 1 ([14]). *Let $\tilde{X} \in \tilde{\mathcal{X}}$ be a fuzzy random variable and let p be a positive probability. Then the value-at-risk $\mathrm{VaR}_p(\tilde{X})$ defined by (15) is a fuzzy number whose α-cuts are*

$$\mathrm{VaR}_p(\tilde{X})_\alpha = [\mathrm{VaR}_p(\tilde{X}_\alpha^-), \mathrm{VaR}_p(\tilde{X}_\alpha^+)] \tag{16}$$

for $\alpha \in (0,1]$.

The value-at-risk (15) on fuzzy random variables has the following properties.

Lemma 2 ([14]). *Let $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$ be fuzzy random variables and let p be a positive probability. Then the value-at-risk VaR_p defined by (15) has the following properties:*

(i) *If $\tilde{X} \preceq \tilde{Y}$, then $\mathrm{VaR}_p(\tilde{X}) \preceq \mathrm{VaR}_p(\tilde{Y})$.*

(ii) $\mathrm{VaR}_p(\zeta \tilde{X}) = \zeta\,\mathrm{VaR}_p(\tilde{X})$ *for $\zeta > 0$.*

(iii) $\mathrm{VaR}_p(\tilde{X} + \tilde{a}) = \mathrm{VaR}_p(\tilde{X}) + \tilde{a}$ *for a fuzzy number $\tilde{a} \in \mathcal{R}$.*

Next since the value-at-risk $\mathrm{VaR}_p(\tilde{R}_t)$ for the rate of return (12) with a portfolio is a fuzzy number, we need to evaluate the fuzziness of fuzzy numbers and fuzzy random variables. There are many studies regarding the evaluation of fuzzy numbers. Two major approaches of them are as follows. One is to use weighting functions([1, 3, 13]) and the other is to use possibility and necessity criteria([4]). Here we adopt the former evaluation method of fuzzy numbers and fuzzy random variables. In the rest of this section we introduce the definitions from [13, 15], and in the next section we estimate the VaR regarding the rate of return (12) by the evaluation method. Yoshida [13] has studied an evaluation of fuzzy numbers by *evaluation weights* which are induced from fuzzy measures to evaluate a confidence degree that a fuzzy number takes values in an interval. With respect to fuzzy random variables, the randomness is evaluated by the probabilistic expectation and the fuzziness is estimated by the evaluation weights and the following function. Let $g^\lambda : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g^\lambda([x,y]) := \lambda x + (1-\lambda) y, \quad [x,y] \in \mathcal{I}, \tag{17}$$

where λ is a constant satisfying $0 \le \lambda \le 1$ and $\mathcal{I}$ denotes the set of all bounded closed intervals. This scalarization is used for the estimation of fuzzy numbers to give a mean value of the interval $[x,y]$ with a weight λ, and g^λ is called a *λ-mean function* and λ is called a *pessimistic-optimistic index* which indicates the pessimistic degree of attitude in decision making ([3]). Let a fuzzy number $\tilde{a} \in \mathcal{R}$. A mean value of the fuzzy number $\tilde{a}$ with respect to λ-mean functions g^λ and an evaluation weight $w(\alpha)$, which depends only on $\tilde{a}$ and α, is given as follows ([15]):

$$\tilde{E}(\tilde{a}) := \frac{\int_0^1 g^\lambda(\tilde{a}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}, \tag{18}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ is the α-cut of the fuzzy number $\tilde{a}$. In (18), $w(\alpha)$ indicates a *confidence degree that the fuzzy*

ISBN: 978-989-95079-6-8

number $\tilde{a}$ takes values in the interval $\tilde{a}_\alpha$ at each level α. Hence, an evaluation weight $w(\alpha)$ is called the *possibility evaluation weight* $w^P(\alpha)$ if $w^P(\alpha) := 1$ for $\alpha \in [0,1]$, and $w(\alpha)$ is called the *necessity evaluation weight* $w^N(\alpha)$ if $w^N(\alpha) := 1-\alpha$ for $\alpha \in [0,1]$. Especially, for a fuzzy number $\tilde{a} \in \mathcal{R}$, the means in the possibility and necessity cases are represented respectively by $\tilde{E}^P(\tilde{a})$ and $\tilde{E}^N(\tilde{a})$, and we consider their combination $\nu\tilde{E}^P(\tilde{a}) + (1-\nu)\tilde{E}^P(\tilde{a})$ with a parameter $\nu \in [0,1]$ ([13, 14, 16]). The mean $\tilde{E}$ has the following natural properties of the linearity and the monotonicity regarding the fuzzy max order.

Lemma 3 ([13, 15]). *Let $\lambda \in [0,1]$. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$ and real numbers θ, ζ, the following (i) – (iv) hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$.

(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$ *if* $\zeta \geq 0$.

(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.

(iv) *If* $\tilde{a} \succeq \tilde{b}$*, then* $\tilde{E}(\tilde{a}) \geq \tilde{E}(\tilde{b})$ *holds.*

For a fuzzy random variable $\tilde{X}$, *the mean of the expectation* $E(\tilde{X})$ is a real number

$$E(\tilde{E}(\tilde{X})) = E\left(\frac{\int_0^1 g^\lambda(\tilde{X}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}\right). \qquad (19)$$

Then, from Lemma 3, we obtain the following results regarding fuzzy random variables.

Lemma 4 ([13, 15]). *Let $\lambda \in [0,1]$. For a fuzzy number $\tilde{a} \in \mathcal{R}$, integrable fuzzy random variables $\tilde{X}, \tilde{Y}$, an integrable real random variable Z and a nonnegative real number ζ, the following (i) – (v) hold.*

(i) $E(\tilde{E}(\tilde{X})) = \tilde{E}(E(\tilde{X}))$.

(ii) $E(\tilde{E}(\tilde{a})) = \tilde{E}(\tilde{a})$ *and* $E(\tilde{E}(Z)) = E(Z)$.

(iii) $E(\tilde{E}(\zeta\tilde{X})) = \zeta E(\tilde{E}(\tilde{X}))$.

(iv) $E(\tilde{E}(\tilde{X} + \tilde{Y})) = E(\tilde{E}(\tilde{X})) + E(\tilde{E}(\tilde{Y}))$.

(v) *If* $\tilde{X} \succeq \tilde{Y}$*, then* $E(\tilde{E}(\tilde{X})) \geq E(\tilde{E}(\tilde{Y}))$ *holds.*

5 The Minimization of the Risk of Falling

In this section, we discuss portfolio problems under uncertainty. First we estimate the rate of return with imprecision for a portfolio. Let the mean, the variance and the covariance of the rate of return R_t^i, which are the real random variables in (2.1), by $\mu_t^i := E(R_t^i)$, $(\sigma_t^i)^2 := E((R_t^i - \mu_t^i)^2)$, and $\sigma_t^{ij} := E((R_t^i - \mu_t^i)(R_t^j - \mu_t^j))$ for $i,j = 1,2,\cdots,n$. Hence we assume that the determinant of the variance-covariance matrix $\Sigma := [\sigma_t^{ij}]$ is not zero and there exists its inverse matrix Σ^{-1}. This assumption is natural and it can be realized easily by taking care of the combinations of assets. For a portfolio $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ $(i = 1,2,\cdots,n)$, we calculate the expectation and the variance regarding $\tilde{R}_t = w^1\tilde{R}_t^1 + w^2\tilde{R}_t^2 + \cdots + w^n\tilde{R}_t^n$. From Lemma 4, the expectation $\tilde{\mu}_t := E(\tilde{E}(\tilde{R}_t))$ follows

$$\tilde{\mu}_t = E(\tilde{E}(\tilde{R}_t)) = \sum_{i=1}^n w^i E(\tilde{E}(\tilde{R}_t^i)) = \sum_{i=1}^n w^i \tilde{\mu}_t^i, \qquad (20)$$

where $\tilde{\mu}_t^i := E(\tilde{E}(\tilde{R}_t^i))$ for $i = 1,2,\cdots,n$. On the other hand, regarding this model, from [13] we can find that the variance $(\tilde{\sigma}_t)^2 := E((\tilde{E}(\tilde{R}_t) - \tilde{\mu}_t)^2)$ of $\tilde{R}_t$ equals to the variance $(\sigma_t)^2 := E((R_t - \mu_t)^2)$ of R_t:

$$(\tilde{\sigma}_t)^2 = (\sigma_t)^2 = \sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^{ij}. \qquad (21)$$

By (14), (20) and (21), the mean of $\mathrm{VaR}_p(\tilde{R}_t)$ is

$$\tilde{E}(\mathrm{VaR}_p(\tilde{R}_t)) = \sum_{i=1}^n w^i \tilde{\mu}_t^i - \kappa\sqrt{\sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^{ij}} \qquad (22)$$

with a positive constant κ. Now step by step we discuss a portfolio problem to minimize the rate of falling $\delta = -\tilde{E}(\mathrm{VaR}_p(\tilde{R}_t))$. First, we deal with a variance-minimizing model. For a given constant γ, which is the minimum expected rate of return to be guaranteed for the portfolio, we consider the following quadratic programming with respect to portfolios with allowance for short selling.

Variance-minimizing problem (P1): Minimize the variance

$$\sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^{ij} \qquad (23)$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ under the following condition regarding the expected rate of return: $\sum_{i=1}^n w^i \tilde{\mu}_t^i = \gamma$.

We start from the following classical results regarding the variance-minimizing problem (P1), and we analyze the portfolio problem to minimize the rate of falling.

Lemma 5 ([15, Theorem 1]). *The solution of the variance-minimizing problem (P1) is given by*

$$w = \xi\Sigma^{-1}\mathbf{1} + \eta\Sigma^{-1}\tilde{\mu} \qquad (24)$$

and then the corresponding variance is

$$\tilde{\rho} := \frac{A\gamma^2 - 2B\gamma + C}{\Delta}, \qquad (25)$$

where $\tilde{\mu}^i := \mu_t^i + \tilde{E}^\lambda(\tilde{a}_t^i)$, $\tilde{\sigma}^{ij} := \sigma_t^{ij}$ $(i,j = 1,2,\cdots,n)$, $\tilde{\Sigma} := [\tilde{\sigma}^{ij}]$, $\tilde{\mu} := [\tilde{\mu}^1\ \tilde{\mu}^2\ \cdots\ \tilde{\mu}^n]^{\mathrm{T}}$, $\mathbf{1} := [1\ 1\ \cdots\ 1]^{\mathrm{T}}$, $\xi := \frac{C - B\gamma}{\Delta}$, $\eta := \frac{A\gamma - B}{\Delta}$, $A := \mathbf{1}^{\mathrm{T}}\tilde{\Sigma}^{-1}\mathbf{1}$, $B := \mathbf{1}^{\mathrm{T}}\tilde{\Sigma}^{-1}\tilde{\mu}$, $C := \tilde{\mu}^{\mathrm{T}}\tilde{\Sigma}^{-1}\tilde{\mu}$, $\Delta := AC - B^2$ *and* T *denotes the transpose of a vector.*

Hence, we consider a risk-sensitive model, which is not of mean-variance types but of mean-standard deviation types, in order to deal with a portfolio problem to minimize the rate of falling in the third step. For a constant γ and a positive constant κ, we discuss the following risk-sensitive portfolio problem with allowance for short selling.

Risk-sensitive problem (P2): Maximize a risk-sensitive expected rate of return

$$v(\gamma) := \sum_{i=1}^n w^i \tilde{\mu}_t^i - \kappa\sqrt{\sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^{ij}} \qquad (26)$$

ISBN: 978-989-95079-6-8

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ $(w^1 + w^2 + \cdots + w^n = 1)$ under the condition $\sum_{i=1}^n w^i \tilde{\mu}_t^i = \gamma$.

Now we discuss the following VaR portfolio problem without allowance for short selling. The following form (27) comes from the value-at-risk $\tilde{E}(\mathrm{VaR}_p(\tilde{R}_t))$ given in (22).

Portfolio problem to minimize the rate of falling (P3):
Minimize the risk of falling

$$\delta = -\sum_{i=1}^n w^i \tilde{\mu}_t^i + \kappa \sqrt{\sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^{ij}} \qquad (27)$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ for $i = 1, 2, \cdots, n$.

Since we have

$$\inf_w (27) = \inf_\gamma \left(\inf_{w:\, \sum_{i=1}^n w^i \tilde{\mu}_t^i = \gamma} (27) \right) = -\sup_\gamma (26),$$

in the same way as [16] we arrive at the following analytical solutions of the portfolio problem to minimize the rate of falling (P3).

Lemma 5. *Let A and Δ be positive. Let κ satisfy $\kappa^2 > C$. The solution of the portfolio problem to minimize the rate of falling (P3) is given by $w^* = \xi\Sigma^{-1}\mathbf{1} + \eta\Sigma^{-1}\tilde{\mu}$, and then the corresponding rate of falling is $\delta(\gamma^*) = -\frac{B-\sqrt{A\kappa^2-\Delta}}{A}$ at the expected rate of return $\gamma^* := \frac{B}{A} + \frac{\Delta}{A\sqrt{A\kappa^2-\Delta}}$, where $\xi := \frac{C-B\gamma^*}{\Delta}$ and $\eta := \frac{A\gamma^*-B}{\Delta}$. Further, if $\Sigma^{-1}\mathbf{1} \geq \mathbf{0}$ and $\Sigma^{-1}\tilde{\mu} \geq \mathbf{0}$, then the portfolio (5.18) satisfies $w^* \geq \mathbf{0}$, i.e. the portfolio w^* is a portfolio without allowance for short selling. Here, $\mathbf{0}$ denotes the zero vector.*

In Lemma 5, we note that the optimal portfolio w^* not only to minimize the rate of falling $\delta(\gamma^*)$ but also to maximize the expected rate of return γ^*.

Theorem 1. *Let A and Δ be positive. Let δ satisfy $\delta > -2B/A$. Then, the assumptions in Lemma 5 are satisfied, and the following (i) and (ii) hold for the optimal portfolio in Lemma 5.*

(i) *For a rate of falling δ, the corresponding constant κ_δ and the expected rate of return γ_δ are given by*

$$\kappa_\delta := \sqrt{A\delta^2 + 2B\delta + C} \quad \text{and} \quad \gamma_\delta := \frac{B\delta + C}{A\delta + B}. \qquad (28)$$

Then the risk probability is $p_\delta = P(\tilde{E}(\tilde{R}_t) \leq -\delta)$.

(ii) *If R_t^i $(i = 1, 2, \cdots, n)$ have normal distributions, the risk probability p_δ in (i) is given by*

$$p_\delta := \Phi(-\kappa_\delta), \qquad (29)$$

where κ_δ is defined by (28) and Φ is the cumulative normal distribution function $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt$ for $z \in \mathbb{R}$.

Remark 1 From Theorem 1, for the optimal portfolio of (P3), the relations among the expected rate of return $\gamma = \gamma_\delta$, the rate of falling δ and the risk probability $p = p_\delta$ are illustrated as follows.

$$\left.\begin{array}{lcl} \gamma & : & \text{the expected rate of return} \\ \Updownarrow & \gamma = & \dfrac{B\delta + C}{A\delta + B} \\ \delta & : & \text{the rate of falling}(= -\mathrm{VaR}) \\ \Updownarrow & p = & \Phi(-\sqrt{A\delta^2 + 2B\delta + C}) \\ p & : & \text{the risk probability of falling} \end{array}\right\} \qquad (30)$$

These results hold for the crisp case ($c_t^i = 0$) and they will be useful not only for analytic study but also for actual management in finance. In the next section, by figures with a numerical example we show the relations (30) among the expected rate of return γ, the rate of falling δ and the risk probability p.

6 A numerical example

In this session, we give a simple example to illustrate our idea. For the numerical computation, we need to evaluate fuzzy numbers representing the rates of return (8). From [13, 14], we have evaluations of the fuzzy numbers $\tilde{a}_t^i$ by a combination $\nu\tilde{E}^P(\tilde{a}) + (1-\nu)\tilde{E}^P(\tilde{a})$ with a parameter $\nu \in [0,1]$ with a parameter $\nu(\in [0,1])$, which is called a *possibility-necessity weight* ([13]). Then from (20) we obtain the expected rate of return

$$\tilde{\mu}_t = \sum_{i=1}^n w^i \left(\mu_t^i + \frac{(1-2\lambda)(4-\nu)}{6} c_t^i \right) \qquad (31)$$

for the possibility-necessity weight $\nu(\in [0,1])$ and the pessimistic-optimistic index $\lambda(\in [0,1])$. In (31), the decision maker may choose the parameters $\lambda(\in [0,1])$ and $\nu(\in [0,1])$. The pessimistic-optimistic index is taken as $\lambda = 1$ if he has pessimistic personal forecast in the market and he takes careful decision, and $\lambda = 0$ if he has optimistic personal forecast and he is not nervous. The possibility-necessity weight is taken as $\nu = 1$ when he has enough confidential information about the market, and $\nu = 0$ when he does not have confidential information. In this model, $\nu = 0$ is reasonable since our objective function is VaR, which is a kind of risk, and we need to take into account of the fuzziness of information in the market. While λ depends on the decision maker's attitude in his investment. In this example, we compute the pessimistic case $\lambda = 1$.

Let $n = 4$ be the number of assets. Take the expected rate of return, a variance-covariance matrix and fuzzy factors as Table 1. We assume that the rate of return R_t^i has the normal distributions. We discuss a risk probability 1% in the normal distribution, and then the corresponding constant is $\kappa = 2.326$, which is given in (14) for VaR. Then, the conditions in Theorem 1 are satisfied and by formulae of Lemma 5 we easily obtain the optimal portfolio $w^* = (w^1, w^2, w^3, w^4) = (0.229604, 0.215551, 0.252000, 0.302845)$ for the portfolio problem to minimize the rate of falling (P3), and then the corresponding rate of falling is $\delta(\gamma^*) = 0.557042$ and the expected rate of return is $\gamma^* = 0.0498026$.

Owing to the equations (28) and (29), we can easily observe the explicit relations among the expected rate of return γ, the rate of falling δ and the risk probability p (Remark 1). Regarding this example, Fig.1 shows the relation among the rate of

ISBN: 978-989-95079-6-8

falling δ, the expected rate of return γ and the risk probability p. Figs.1 and 2 are drawn based on Remark 1. Especially Fig.2 shows the relation among the risk probability p such that the falling in the asset prices will be lower than $100(1-\delta)\,\%$, the rate of falling δ and the expected rate of return γ. For example, for a rate of falling $\delta = 0.2$, which implies that the asset price will fall lower than $80\,\%$, its risk probability is $p = 0.168993$, however we have the expected rate of return $\gamma = 0.0504023$ with $\kappa = 0.958154$, which are calculated easily from Remark 1 based on Theorem 1. We can also compute the cases of falling below par value and the complete bankruptcy, taking $\delta = 0$ and $\delta = 1$ respectively. The probability of falling below par value is $p_0 = 0.4211443$ and then the expected rate of return $\gamma_0 = 0.0545276$, and the probability of complete bankruptcy is $p_1 = 0.0000285992$ and then the expected rate of return $\gamma_1 = 0.0496258$.

Table 1: Expected rates of return, a variance-covariance matrix and fuzzy factors.

Asset	μ_t^i
w_1	0.05
w_2	0.07
w_3	0.06
w_4	0.04

Asset	c_t^i
w_1	0.006
w_2	0.008
w_3	0.007
w_4	0.005

σ_t^{ij}	w_1	w_2	w_3	w_4
w_1	0.35	0.03	0.02	-0.08
w_2	0.03	0.25	-0.06	0.08
w_3	0.02	-0.06	0.33	-0.02
w_4	-0.08	0.08	-0.02	0.24

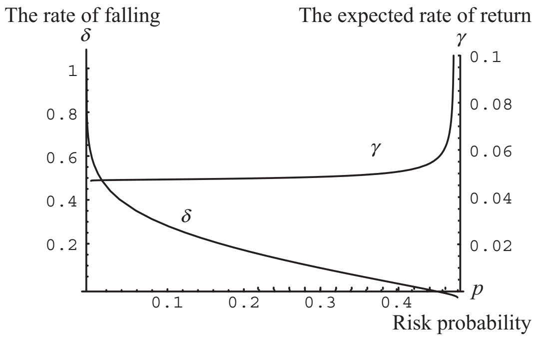

Figure 1: The rate of falling δ and the expected rate of return γ at risk probabilities p.

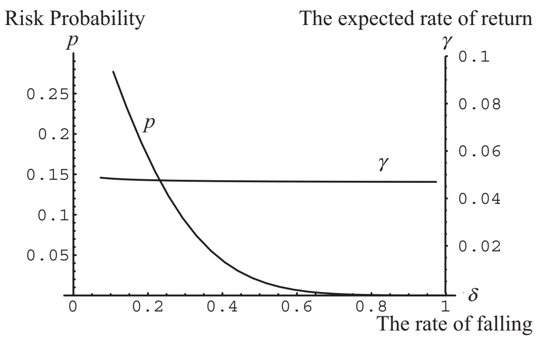

Figure 2: Risk probability p and the expected rate of return γ at the rates of falling δ.

7 Conclusion

We have derived the relation (30) among the expected rate of return γ, the rate of falling δ and the risk probability p. The relation holds for the crisp case and it will be useful not only for analytic study but also for actual management in finance.

References

[1] L.Campos and A.Munoz, A subjective approach for ranking fuzzy numbers, *Fuzzy Sets and Systems*, 29:145-153, 1989.

[2] L.El Chaoui, M.Oks and F.Oustry, Worst-case value at risk and robust portfolio optimization: A conic programming approach, *Operations Research*, 51:543-556, 2003.

[3] P. Fortemps and M. Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems*, 82:319-330, 1996.

[4] M.Inuiguchi and T.Tanino, Portfolio selection under independent possibilistic information, *Fuzzy Sets and Systems*, 115:83-92, 2000.

[5] P.Jorion, *Value at Risk: Tne New Benchmark for Managing Financial Risk*, Third Edition McGraw-Hill, New York, 2007.

[6] R.Kruse and K.D.Meyer, *Statistics with Vague Data*, Riedel Publ. Co., Dortrecht, 1987.

[7] H.Kwakernaak, Fuzzy random variables – I. Definitions and theorem, *Inform. Sci.*, 15:1-29, 1978.

[8] H.Markowitz, *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, Blackwell, Oxford, 1990.

[9] S.R.Pliska, *Introduction to Mathematical Finance: Discrete-Time Models*, Blackwell Publ., New York, 1997.

[10] M.L.Puri and D.A.Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.*, 114:409-422, 1986.

[11] M.C.Steinbach, Markowitz revisited: Mean-variance model in financial portfolio analysis, *SIAM Review*, 43:31-85, 2001.

[12] H.Tanaka, P.Guo and L.B.Türksen, Portfolio selection based on fuzzy probabilities and possibilistic distribution, *Fuzzy Sets and Systems*, 111:397-397, 2000.

[13] Y.Yoshida, Mean values, measurement of fuzziness and variance of fuzzy random variables for fuzzy optimization, *Proceedings of SCIS & ISIS 2006*, 2277-2282, Sept. 2006.

[14] Y.Yoshida, A risk-minimizing model under uncertainty in portfolio, in: V.Torra et al. eds., *MDAI 2007*, LNAI 4529, 295-306, Springer, August, 2007.

[15] Y.Yoshida, Fuzzy extension of estimations with randomness: The perception-based approach, in: P.Melin et al. eds., *IFSA2007*, LNAI 4617, 381-391, Springer, Sept., 2007.

[16] Y.Yoshida, An estimation model of value-at-risk portfolio under uncertainty, to appear in *Fuzzy Sets and Systems*, doi:10.1016/j.fss2009.02.007.

[17] Z.Zmeškal, Value at risk methodology of international index portfolio under soft conditions (fuzzy-stochastic approach), *International Review of Financial Analysis*, 14:263-275, 2005.

[18] L.A.Zadeh, Fuzzy sets, *Inform. and Control*, 8:338-353, 1965.

ISBN: 978-989-95079-6-8