Approximations by interval, triangular and trapezoidal fuzzy numbers

Chi-Tsuen Yeh

Department of Mathematics Education, National University of Tainan
33, Sec. 2, Shu-Lin St., Tainan city 70005, Taiwan
Email: ctyeh@mail.nutn.edu.tw

Abstract— *Recently, many scholars investigated interval, triangular, and trapezoidal approximations of fuzzy numbers. These researches can be grouped into two classes: the Euclidean distance class and the non-Euclidean distance class. Most approximations in the Euclidean distance class can be calculated by formulas, but calculating approximations in the other class is more complicated. In this paper, we study interval, triangular, and trapezoidal approximations under a weighted Euclidean distance which generalize all approximations in the Euclidean distance class. First, we embed fuzzy numbers into a Hilbert space, and then introduce these weighted approximations by means of best approximations from closed convex subsets of the Hilbert space. Finally, we apply the reduction principle to simplify calculations of these approximations.*

Keywords— weighted trapezoidal approximation, triangular fuzzy number, Hilbert space

1 Introduction

In practice, fuzzy intervals are often used to represent uncertain or incomplete information. An interesting problem is to approximate general fuzzy intervals by interval, triangular, and trapezoidal fuzzy numbers, so as to simplify calculations. Recently, many scholars investigated these approximations of fuzzy numbers. According to the different aspects of distance, these researches can be grouped into two classes: the Euclidean distance class and the non-Euclidean distance class. The Euclidean distance class includes the interval approximation (proposed by Grzegorzewski in 2002 [11]), symmetric triangular approximation (proposed by Ma et al. in 2000 [18]), trapezoidal approximation (proposed by Abbasbandy and Asady in 2004 [1]), and weighted triangular approximation (proposed by Zeng and Li in 2007 [23]). The non-Euclidean distance class includes the rectangle approximation under the Hamming distance (proposed by Chanas in 2001 [6]), symmetric and non-symmetrical trapezoidal approximations under the Euclidean distance between the respective 1/2-levels (proposed by Delgado et al. in 1998 [7]), and trapezoidal approximation under the source distance (proposed by Abbasbandy and Amirfakhrian in 2006 [2]). Some other approximations are also investigated, such as the nearest parametric approximation (proposed by Nasibova and Peker in 2008 [19]), trapezoidal approximation preserving the expected interval (proposed by Grzegorzewski and Mrówka [12, 13, 14], and improved by Ban [5] and Yeh [22] in 2008, independently), approximation by π functions (proposed by Hassine et al. in 2006 [15]), and polynomial approximation (proposed by Abbasbandy and Amirfakhrian in 2006 [3]). Most approximations in the Euclidean distance class can be calculated by formulas, but calculating the approximations in the other class is more complicated. In this paper, we study interval, triangular, and trapezoidal approximations under a weighted Euclidean distance which generalize all approximations in the Euclidean distance class. In Section 2, we define a weighted L^2-distance on space of fuzzy numbers, and then embed the space into the Hilbert space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$ by applying the weighted L^2-distance. In Section 3, we introduce weighted approximations of fuzzy numbers by means of best approximations from closed convex subsets of the Hilbert space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$. Some preliminaries are presented. In Section 4-6, by applying the reduction principle [8, p.80] we compute straightforwardly these approximations of fuzzy numbers, and then propose several important theorems.

2 Embedding fuzzy numbers into the Hilbert space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$

By an *inner product space* we mean that it is a (real) vector space V equipped with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ obeying the following axioms:

1. $\langle u, u \rangle \geq 0$ for all $u \in V$, and $\langle u, u \rangle = 0$ iff (if and only if) $u = 0$,
2. $\langle u, v \rangle = \langle v, u \rangle$, for all $u, v \in V$,
3. $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$, for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$.

An inner product is a metric space if the distance is defined by

$$\mathrm{d}(u, v) := \langle u - v, u - v \rangle^{\frac{1}{2}}.$$

A completely inner product space is often called a *Hilbert space*. It is well-known that the set of all L^2-integrable functions is a Hilbert space, denoted by $L^2_\lambda[0,1]$, on which the inner product is defined as

$$\langle f, g \rangle_\lambda := \int_0^1 f(t)g(t)\lambda(t)dt,$$

where $\lambda = \lambda(t)$ is a nonnegative function on $[0,1]$ with $\int_0^1 \lambda(t)dt > 0$.

Another important Hilbert space is the product space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$, which will be discussed in this paper. Its inner product is defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_\lambda := \int_0^1 [f_1(t)g_1(t) + f_2(t)g_2(t)]\, \lambda(t)dt$$

ISBN: 978-989-95079-6-8

for all (f_1, f_2) and $(g_1, g_2) \in L^2_\lambda[0,1] \times L^2_\lambda[0,1]$. We hence obtain

$$\begin{aligned} &\mathrm{d}^2_\lambda((f_1, f_2), (g_1, g_2)) \\ =&\langle (f_1 - g_1, f_2 - g_2), (f_1 - g_1, f_2 - g_2)\rangle_\lambda \\ =&\int_0^1 \left(|f_1(t) - g_1(t)|^2 + |f_2(t) - g_2(t)|^2\right)\lambda(t)dt. \end{aligned}$$

Recall that, a fuzzy number $\tilde{A}$ can be represented by an ordered pair of left continuous functions $[A_L(\alpha), A_U(\alpha)]$ (called the α-cuts of $\tilde{A}$), $0 \le \alpha \le 1$, which satisfy the following conditions: (1) A_L is increasing on [0,1], (2) A_U is decreasing on [0,1], (3) $A_L(1) \le A_U(1)$. Let $\tilde{\mathbb{F}}$ denote the set of all fuzzy numbers. The weighted L^2-*distance* (*Euclidean distance*) on $\tilde{\mathbb{F}}$ is defined as

$$\begin{aligned} \mathrm{d}_\lambda(\tilde{A}, \tilde{B}) :=\Big[&\int_0^1 |A_L(\alpha) - B_L(\alpha)|^2 \lambda(\alpha) d\alpha \\ &+ \int_0^1 |A_U(\alpha) - B_U(\alpha)|^2 \lambda(\alpha) d\alpha\Big]^{\frac{1}{2}}. \end{aligned} \tag{1}$$

For more generality, we refer to [10] in which Grzegorzewski proposed two families of general distances on $\tilde{\mathbb{F}}$. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. The fuzzy addition and fuzzy subtraction operations on $\tilde{\mathbb{F}}$ are defined as follows:

$$\begin{aligned} \tilde{A} + \tilde{B} &:= [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)], \\ \tilde{A} - \tilde{B} &:= [A_L(\alpha) - B_U(\alpha), A_U(\alpha) - B_L(\alpha)]. \end{aligned}$$

The above conditions (1)-(3) (the definition of fuzzy numbers) imply that A_L and $A_U \in L^2_\lambda[0,1]$, hence we define

$$i : \tilde{A} \mapsto (A_L, A_U) \in L^2_\lambda[0,1] \times L^2_\lambda[0,1].$$

In the following, we always use the interval notation $[A_L, A_U]$ instead of (A_L, A_U), although it may make little sense. Notice that, the fuzzy addition operation coincides with the vector addition on $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$ and its inverse operation (vector subtraction) is not the fuzzy subtraction "-". Let the symbol "$\ominus$" denote the inverse operation, that is

$$\tilde{A} \ominus \tilde{B} := [A_L(\alpha) - B_L(\alpha), A_U(\alpha) - B_U(\alpha)],$$

which is often called the Hukuhara difference, see [17]. In fact, $\tilde{A} \ominus \tilde{B}$ may be not in $\tilde{\mathbb{F}}$. From Eq.(1), we find that

$$\mathrm{d}^2_\lambda(\tilde{A}, \tilde{B}) = \langle \tilde{A} \ominus \tilde{B}, \tilde{A} \ominus \tilde{B}\rangle_\lambda.$$

This shows that we can embed space of fuzzy numbers into the Hilbert space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$. We hence define an inner product on $\tilde{\mathbb{F}}$ inheriting from $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$, that is

$$\langle \tilde{A}, \tilde{B}\rangle_\lambda := \int_0^1 [A_L(\alpha)B_L(\alpha) + A_U(\alpha)B_U(\alpha)]\, \lambda(\alpha) d\alpha. \tag{2}$$

3 Approximations of fuzzy numbers

Let Ω be a subset of a Hilbert space V, then we call that:

1. Ω is a *subspace* iff $u + v \in \Omega$ and $ru \in \Omega$ for all $u, v \in \Omega$ and all $r \in \mathbb{R}$,
2. Ω is *convex* iff $ru + (1-r)v \in \Omega$ for all $u, v \in \Omega$ and all $r \in [0,1]$,
3. Ω is *chebyshev* iff for each $u \in V$ there exists a unique element $P_\Omega(u) \in \Omega$ such that

   $$\mathrm{d}(u, P_\Omega(u)) \le \mathrm{d}(u, x), \quad \forall x \in \Omega,$$

   and then $P_\Omega(u)$ is called the *best approximation* of u from Ω.

It is well-known that every closed convex subset (closed subspace, finite dimensional subspace) is chebyshev, see [8, p.23-24]. For any closed convex subset Ω, there is a sufficient and necessary condition of the best approximation $P_\Omega(u)$, as follows

$$\langle u - P_\Omega(u), x - P_\Omega(u)\rangle \le 0, \quad \forall x \in \Omega.$$

Furthermore, we also have

$$\mathrm{d}(P_\Omega(u), P_\Omega(v)) \le \mathrm{d}(u, v), \quad \forall u, v \in V, \tag{3}$$

refer to [21, Appendix C]. This implies P_Ω is continuous. While Ω is a closed subspace, then the above condition becomes

$$\langle u - P_\Omega(u), x\rangle = 0, \quad \forall x \in \Omega,$$

or equivalently

$$\langle u, x\rangle = \langle P_\Omega(u), x\rangle, \quad \forall x \in \Omega. \tag{4}$$

In this paper, all elements in $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$ of the form

$$[r_1 + (r_2 - r_1)\alpha, r_4 - (r_4 - r_3)\alpha]$$

are called *extended trapezoidal* fuzzy numbers. Let $\mathbb{T}$ denote the subset of all extended trapezoidal fuzzy numbers. It is easy to see that, an element $\tilde{A} = [r_1 + (r_2 - r_1)\alpha, r_4 - (r_4 - r_3)\alpha]$ is *trapezoidal* iff $\tilde{A} \in \tilde{\mathbb{F}} \cap \mathbb{T}$, that is

$$r_1 \le r_2 \le r_3 \le r_4. \tag{5}$$

Also, a trapezoidal fuzzy number $\tilde{A}$ is *triangular* (resp. *symmetric trapezoidal*, *symmetric triangular*, *interval*) iff $r_2 = r_3$ (resp. $r_2 - r_1 = r_4 - r_3$, $r_2 = r_3$ and $r_2 - r_1 = r_4 - r_3$, $r_1 = r_2$ and $r_3 = r_4$). Let $\tilde{\mathbb{T}}$, $\tilde{\mathbb{T}}_s$, $\tilde{\Delta}$, $\tilde{\Delta}_s$, and $\tilde{\mathbb{I}}$ denote the sets of all trapezoidal, symmetric trapezoidal, triangular, symmetric triangular, and interval fuzzy numbers, respectively. It is easy to verify that

1. $\mathbb{T}$ is a closed subspace of $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$, and
2. $\tilde{\mathbb{F}}$, $\tilde{\mathbb{T}}$, $\tilde{\mathbb{T}}_s$, $\tilde{\Delta}$, $\tilde{\Delta}_s$, and $\tilde{\mathbb{I}}$ are all closed convex subsets.

Hence, all of them are chebyshev. The best approximations of u from $\mathbb{T}$, $\tilde{\mathbb{T}}$, $\tilde{\mathbb{T}}_s$, $\tilde{\Delta}$, $\tilde{\Delta}_s$, and $\tilde{\mathbb{I}}$ are called the *extended trapezoidal*, *trapezoidal*, *symmetric trapezoidal*, *triangular*, *symmetric triangular*, and *interval approximations* of u, respectively. Eq.(3) implies that these approximations are continuous.

Theorem 1 (The reduction principle [8, p.80]). *Let K be a closed convex subset of an inner product space V and M be any chebyshev subspace of V that contains K. Then, we have that*

$$P_K(u) = P_K(P_M(u)) \quad \textit{and}$$
$$\mathrm{d}(u, P_K(u))^2 = \mathrm{d}(u, P_M(u))^2 + \mathrm{d}(P_M(u), P_K(u))^2.$$

ISBN: 978-989-95079-6-8

Now, let's define four extended trapezoidal fuzzy numbers:

$$\tilde{E}_1 := [1-\alpha, 0], \qquad \tilde{E}_2 := [\alpha, 0],$$
$$\tilde{E}_3 := [0, \alpha], \qquad \tilde{E}_4 := [0, 1-\alpha].$$

Then, each element in $\mathbb{T}$ is a linear combination of $\tilde{E}_i$, $1 \le i \le 4$, for instance

$$[r_1 + (r_2 - r_1)\alpha, r_4 - (r_4 - r_3)\alpha] = \sum_{i=1}^{4} r_i \tilde{E}_i.$$

This implies $\mathbb{T} = \text{Span}\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\}$. We also define two other subspaces of $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$ as follows

$$\Delta := \text{Span}\{\tilde{E}_1, \tilde{E}_2 + \tilde{E}_3, \tilde{E}_4\},$$
$$\mathbb{I} := \text{Span}\{\tilde{E}_1 + \tilde{E}_2, \tilde{E}_3 + \tilde{E}_4\} = \text{Span}\{[1,0],[0,1]\}.$$

It is easy to see that $\tilde{\mathbb{I}} \subset \mathbb{I}$, $\tilde{\Delta}_s \subset \tilde{\Delta} \subset \Delta$, and $\tilde{\mathbb{T}}_s \subset \tilde{\mathbb{T}} \subset \mathbb{T}$. By applying the reduction principle, we obtain that

$$P_{\tilde{\mathbb{I}}}(u) = P_{\tilde{\mathbb{I}}}(P_{\mathbb{I}}(u)), \tag{6}$$
$$P_{\tilde{\Delta}}(u) = P_{\tilde{\Delta}}(P_\Delta(u)), \quad P_{\tilde{\Delta}_s}(u) = P_{\tilde{\Delta}_s}(P_\Delta(u)), \tag{7}$$
$$P_{\tilde{\mathbb{T}}}(u) = P_{\tilde{\mathbb{T}}}(P_{\mathbb{T}}(u)), \quad P_{\tilde{\mathbb{T}}_s}(u) = P_{\tilde{\mathbb{T}}_s}(P_{\mathbb{T}}(u)). \tag{8}$$

4 The interval approximations

In 2002, Grzegorzewski first proposed interval approximations of fuzzy numbers [11]. Let's extend his results to the case of weighted L^2-distance. We now start with computing the best approximation $P_{\mathbb{I}}(\tilde{A})$ of any fuzzy number $\tilde{A} = [A_L(\alpha), A_U(\alpha)]$ from the subspace $\mathbb{I}$. Unless otherwise stated, we fix the following real numbers:

$$\lambda_0 := \int_0^1 \lambda(\alpha) d\alpha > 0$$

and

$$L_0 := \int_0^1 A_L(\alpha)\lambda(\alpha)d\alpha, \quad U_0 := \int_0^1 A_U(\alpha)\lambda(\alpha)d\alpha.$$

From Eq.(2), we find that $\langle[1,0],[0,1]\rangle_\lambda = 0$ and

$$\langle[1,0],[1,0]\rangle_\lambda = \langle[0,1],[0,1]\rangle_\lambda = \lambda_0.$$

Since $\mathbb{I} = \text{Span}\{[1,0],[0,1]\}$, we may assume that

$$P_{\mathbb{I}}(\tilde{A}) = r[1,0] + s[0,1].$$

By applying Eq.(4), we can solve

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \langle[1,0],[1,0]\rangle_\lambda & \langle[0,1],[1,0]\rangle_\lambda \\ \langle[1,0],[0,1]\rangle_\lambda & \langle[0,1],[0,1]\rangle_\lambda \end{pmatrix}^{-1} \begin{pmatrix} \langle\tilde{A},[1,0]\rangle_\lambda \\ \langle\tilde{A},[0,1]\rangle_\lambda \end{pmatrix}$$
$$= \lambda_0^{-1} \begin{pmatrix} L_0 \\ U_0 \end{pmatrix}.$$

Hence, we obtain

$$P_{\mathbb{I}}(\tilde{A}) = [\lambda_0^{-1} L_0, \lambda_0^{-1} U_0].$$

The fact $A_L(\alpha) \le A_U(\alpha)$ implies $L_0 \le U_0$, hence $P_{\mathbb{I}}(\tilde{A}) \in \tilde{\mathbb{I}}$. By applying Eq.(6), we obtain the following theorem.

Theorem 2. *Let $\tilde{A}$ be a fuzzy number. Then, its interval approximation is $P_{\tilde{\mathbb{I}}}(\tilde{A}) = [\lambda_0^{-1} L_0, \lambda_0^{-1} U_0]$.*

While $\lambda(\alpha) = 1$, we get that $\lambda_0 = 1$. Then, the above equation coincides with the Grzegorzewski's formula [11, Equations (15) and (16)]. Also, the interval $[L_0, U_0]$ is called the *expected interval* of $\tilde{A}$, which is introduced by Dubois and Prade [9] and Heilpern [16], independently.

5 The triangular approximations

Let $\lambda = \lambda(\alpha)$ be a nonnegative function on [0,1] with $\int_0^1 \lambda(\alpha)\, d\alpha > 0$. In what follows, we fix

$$a := \int_0^1 (1-\alpha)^2 \lambda(\alpha) d\alpha > 0,$$
$$b := \int_0^1 \alpha(1-\alpha)\lambda(\alpha) d\alpha > 0,$$
$$c := \int_0^1 \alpha^2 \lambda(\alpha) d\alpha > 0,$$

and

$$L_1 := \int_0^1 A_L(\alpha)\alpha\lambda(\alpha)d\alpha, \quad U_1 := \int_0^1 A_U(\alpha)\alpha\lambda(\alpha)d\alpha.$$

By applying Schwarz inequality, we get

$$ac - b^2 > 0.$$

In a similar manner, we start with computing $P_\Delta(\tilde{A})$. Recall that $\Delta := \text{Span}\{\tilde{E}_1, \tilde{E}_2 + \tilde{E}_3, \tilde{E}_4\}$, so we compute

$$\begin{pmatrix} \langle\tilde{E}_1,\tilde{E}_1\rangle_\lambda & \langle\tilde{E}_2+\tilde{E}_3,\tilde{E}_1\rangle_\lambda & \langle\tilde{E}_4,\tilde{E}_1\rangle_\lambda \\ \langle\tilde{E}_1,\tilde{E}_2+\tilde{E}_3\rangle_\lambda & \langle\tilde{E}_2+\tilde{E}_3,\tilde{E}_2+\tilde{E}_3\rangle_\lambda & \langle\tilde{E}_4,\tilde{E}_2+\tilde{E}_3\rangle_\lambda \\ \langle\tilde{E}_1,\tilde{E}_4\rangle_\lambda & \langle\tilde{E}_2+\tilde{E}_3,\tilde{E}_4\rangle_\lambda & \langle\tilde{E}_4,\tilde{E}_4\rangle_\lambda \end{pmatrix}$$
$$= \begin{pmatrix} a & b & 0 \\ b & 2c & b \\ 0 & b & a \end{pmatrix}.$$

Let $P_\Delta(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4$. By applying Eq.(4), we can solve

$$\begin{pmatrix} r_1 \\ r_2 \\ r_4 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ b & 2c & b \\ 0 & b & a \end{pmatrix}^{-1} \begin{pmatrix} \langle\tilde{A},\tilde{E}_1\rangle_\lambda \\ \langle\tilde{A},\tilde{E}_2+\tilde{E}_3\rangle_\lambda \\ \langle\tilde{A},\tilde{E}_4\rangle_\lambda \end{pmatrix}$$
$$= \frac{1}{\delta} \begin{pmatrix} 2ac-b^2 & -ab & b^2 \\ -ab & a^2 & -ab \\ b^2 & -ab & 2ac-b^2 \end{pmatrix} \begin{pmatrix} L_0 - L_1 \\ L_1 + U_1 \\ U_0 - U_1 \end{pmatrix}. \tag{9}$$

where $\delta = 2a(ac - b^2) > 0$. If we assume $\int_0^1 \lambda(\alpha)d\alpha = \frac{1}{2}$ (that is $a + 2b + c = \frac{1}{2}$) additionally, then the above $P_\Delta(\tilde{A})$ is equal to Zeng and Li's weighted triangular approximation [23]. Notice that $P_\Delta(\tilde{A})$ may be not in $\mathbb{F}$, refer to [21]. That shows that $P_{\tilde{\Delta}}(\tilde{A}) \ne P_\Delta(\tilde{A})$.

Lemma 3. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_\Delta(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4,$$

where r_1, r_2, and r_4 are computed by Eq.(9). Then,

1. *$r_1 \le r_4$,*
2. *if $r_2 \le r_1$, then*
$$-(a+b)L_0 + (a+3b+2c)U_0 - 2(a+2b+c)U_1 \ge 0,$$
3. *if $r_2 \ge r_4$, then*
$$-(a+3b+2c)L_0 + 2(a+2b+c)L_1 + (a+b)U_0 \ge 0.$$

ISBN: 978-989-95079-6-8

Proof. Omitted.

From Eq.(7), we find $P_{\tilde{\Delta}}(\tilde{A}) = P_{\tilde{\Delta}}(P_{\Delta}(\tilde{A}))$. Eq.(5) implies that an element

$$r\tilde{E}_1 + s(\tilde{E}_2 + \tilde{E}_3) + t\tilde{E}_4 \in \Delta$$

is triangular iff $r \leq s \leq t$. By Lemma 3.1, we obtain that $P_{\Delta}(\tilde{A}) \notin \tilde{\Delta}$ implies either $r_2 < r_1$ or $r_2 > r_4$, hence the best approximation (triangular approximation)

$$P_{\tilde{\Delta}}(\tilde{A}) := r_1'\tilde{E}_1 + r_2'(\tilde{E}_2 + \tilde{E}_3) + r_4'\tilde{E}_4$$

will satisfy $r_2' = r_1'$ or $r_2' = r_4'$, respectively. For instance, suppose that the fuzzy number $\tilde{A}$ has the approximation

$$P_{\Delta}(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4 \quad \text{with } r_2 < r_1.$$

Then, $P_{\tilde{\Delta}}(\tilde{A})$ will belong to Span $\{\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3, \tilde{E}_4\}$, since $r_2' = r_1'$. So, we consider the best approximation of $\tilde{A}$ from Span $\{\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3, \tilde{E}_4\}$. We hence let

$$P_{\tilde{\Delta}}(\tilde{A}) = r_1'(\tilde{E}_1+\tilde{E}_2+\tilde{E}_3)+r_4'\tilde{E}_4 = r_1'[1,\alpha]+r_4'[0,1-\alpha].$$

By applying Eq.(4), we can solve

$$\begin{aligned}\begin{pmatrix} r_1' \\ r_4' \end{pmatrix} &= \begin{pmatrix} a+2b+2c & b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, [1,\alpha]\rangle_\lambda \\ \langle \tilde{A}, [0,1-\alpha]\rangle_\lambda \end{pmatrix} \\ &= \frac{1}{\delta}\begin{pmatrix} a & -b \\ -b & a+2b+2c \end{pmatrix}\begin{pmatrix} L_0+U_1 \\ U_0-U_1 \end{pmatrix},\end{aligned} \tag{10}$$

where $\delta = (a+b)^2 + 2(ac-b^2)$, and we have substituted by

$$\langle [1,\alpha],[1,\alpha]\rangle_\lambda = \int_0^1 (1+\alpha^2)\lambda(\alpha)d\alpha = a+2b+2c.$$

Notice that, the extended trapezoidal fuzzy number

$$\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 = [1,\alpha]$$

does not belong to $\tilde{\mathbb{F}}$. But, this does not effect results of our computation. Applying Lemma 3.2, the reader can easily verify $r_1' \leq r_4'$ in Eq.(10), so that

$$r_1'[1-\alpha,0] + r_4'[\alpha,1] \in \tilde{\mathbb{F}}.$$

Hence, we obtain

$$P_{\tilde{\Delta}}(\tilde{A}) = r_1'[1,\alpha] + r_4'[0,1-\alpha],$$

where r_1' and r_4' are computed by Eq.(10).

On the other hand, if $\tilde{A}$ has the approximation

$$P_{\Delta}(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4 \quad \text{with } r_2 > r_4$$

then its triangular approximation leads to

$$P_{\tilde{\Delta}}(\tilde{A}) = r_1'[1-\alpha,0] + r_4'[\alpha,1],$$

where

$$\begin{aligned}\begin{pmatrix} r_1' \\ r_4' \end{pmatrix} &= \begin{pmatrix} a & b \\ b & a+2b+2c \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, [1-\alpha,0]\rangle_\lambda \\ \langle \tilde{A}, [\alpha,1]\rangle_\lambda \end{pmatrix} \\ &= \frac{1}{\delta}\begin{pmatrix} a+2b+2c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} L_0-L_1 \\ L_1+U_0 \end{pmatrix}.\end{aligned} \tag{11}$$

where $\delta = (a+b)^2 + 2(ac-b^2)$. Again, applying Lemma 3.3 the reader can easily verify $r_1' \leq r_4'$ in Eq.(11).

Theorem 4. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_{\Delta}(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4,$$

where r_1, r_2 and r_4 are computed by Eq.(9). Then, the triangular approximation $P_{\tilde{\Delta}}(\tilde{A})$ can be determined in the following cases:

1. *If $r_1 \leq r_2 \leq r_4$, then*
$$P_{\tilde{\Delta}}(\tilde{A}) = [r_1 + (r_2 - r_1)\alpha, r_4 - (r_4 - r_2)\alpha].$$
2. *If $r_2 < r_1$, then $P_{\tilde{\Delta}}(\tilde{A}) = [r_1', r_4' - (r_4' - r_1')\alpha]$, where r_1' and r_4' are computed by Eq.(10).*
3. *If $r_2 > r_4$, then $P_{\tilde{\Delta}}(\tilde{A}) = [r_1' + (r_4' - r_1')\alpha, r_4']$, where r_1' and r_4' are computed by Eq.(11).*

Next, we compute the symmetric triangular approximation $P_{\tilde{\Delta}_s}(\tilde{A})$ which was first proposed by Ma et al. [18]. Recall that, an extended trapezoidal element

$$r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4 \in \Delta$$

is symmetric triangular iff

$$r_2 - r_1 = r_4 - r_2 \geq 0.$$

Hence, by substituting $r_2 = \frac{1}{2}(r_1 + r_4)$ we get

$$r_1\tilde{E}_1+r_2(\tilde{E}_2+\tilde{E}_3)+r_4\tilde{E}_4 = r_1[1-\frac{1}{2}\alpha, \frac{1}{2}\alpha]+r_4[\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha].$$

Let's consider the best approximation $P_{\Delta_s}(\tilde{A})$ of $\tilde{A}$ from the subspace Δ_s, where Δ_s is defined by

$$\Delta_s := \text{Span } \{[1-\frac{1}{2}\alpha, \frac{1}{2}\alpha], [\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha]\}.$$

Suppose that

$$P_{\Delta_s}(\tilde{A}) = r_1[1-\frac{1}{2}\alpha, \frac{1}{2}\alpha] + r_4[\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha].$$

By applying Eq.(4), we can solve

$$\begin{aligned}\begin{pmatrix} r_1 \\ r_4 \end{pmatrix} &= \begin{pmatrix} a+b+\frac{c}{2} & b+\frac{c}{2} \\ b+\frac{c}{2} & a+b+\frac{c}{2} \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, [1-\frac{1}{2}\alpha, \frac{1}{2}\alpha]\rangle_\lambda \\ \langle \tilde{A}, [\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha]\rangle_\lambda \end{pmatrix} \\ &= \frac{1}{\delta}\begin{pmatrix} a+b+\frac{c}{2} & -(b+\frac{c}{2}) \\ -(b+\frac{c}{2}) & a+b+\frac{c}{2} \end{pmatrix}\begin{pmatrix} L_0 - \frac{1}{2}L_1 + \frac{1}{2}U_1 \\ \frac{1}{2}L_1 + U_0 - \frac{1}{2}U_1 \end{pmatrix},\end{aligned} \tag{12}$$

where $\delta = a^2 + 2ab + ac$. The reader can verify that $r_1 \leq r_4$ for any $\tilde{A} \in \tilde{\mathbb{F}}$. This implies $P_{\Delta_s}(\tilde{A}) \in \tilde{\mathbb{F}}$, so that

$$P_{\tilde{\Delta}_s}(\tilde{A}) = P_{\Delta_s}(\tilde{A}).$$

Theorem 5. *Let $\tilde{A}$ be a fuzzy number. Then, its symmetric approximation is*

$$P_{\tilde{\Delta}_s}(\tilde{A}) = [r_1 + \frac{r_4 - r_1}{2}\alpha, r_4 - \frac{r_4 - r_1}{2}\alpha],$$

where r_1 and r_4 are computed by Eq.(12).

While $\lambda(\alpha) = 1$, we get that, $a = \frac{1}{3}$, $b = \frac{1}{6}$, and $c = \frac{1}{3}$. Substituting into the above equation, we obtain

$$P_{\tilde{\Delta}_s}(\tilde{A}) = [x_0 - \sigma(1-\alpha), x_0 + \sigma(1-\alpha)],$$

where $x_0 = \frac{L_0+U_0}{2}$ and $\sigma = \frac{3}{2}(-L_0 + L_1 + U_0 - U_1)$. This formula coincides with [18, Equations (8) and (9)].

ISBN: 978-989-95079-6-8

6 The trapezoidal approximations

The (extended) trapezoidal approximation was first proposed by Abbasbandy and Asady in 2004 [1]. Afterwards, Grzegorzewski and Mrówka proposed the (extended) trapezoidal approximation preserving the expected interval [12]. Since the expected interval of any fuzzy number $\tilde{A}$ is equal to $P_{\mathbb{I}}(\tilde{A})$ and $\mathbb{I} \supseteq \mathbb{T} \supseteq \tilde{\mathbb{T}}$, by the reduction principle we get that these two (extended) trapezoidal approximations are equal. Now, we start with computing the extended trapezoidal approximation $P_{\mathbb{T}}(\tilde{A})$.

Because that $\mathbb{T} = \text{Span}\,\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\}$, we let

$$P_{\mathbb{T}}(\tilde{A}) = t_1\tilde{E}_1 + t_2\tilde{E}_2 + t_3\tilde{E}_3 + t_4\tilde{E}_4.$$

In the same vein, by applying Eq.(4) we can solve

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & b & a \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, \tilde{E}_1 \rangle_\lambda \\ \langle \tilde{A}, \tilde{E}_2 \rangle_\lambda \\ \langle \tilde{A}, \tilde{E}_3 \rangle_\lambda \\ \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda \end{pmatrix}$$
$$= \frac{1}{ac - b^2} \begin{pmatrix} c & -b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -b & c \end{pmatrix} \begin{pmatrix} L_0 - L_1 \\ L_1 \\ U_1 \\ U_0 - U_1 \end{pmatrix} \tag{13}$$
$$= \frac{1}{ac - b^2} \begin{pmatrix} cL_0 - (b+c)L_1 \\ -bL_0 + (a+b)L_1 \\ -bU_0 + (a+b)U_1 \\ cU_0 - (b+c)U_1 \end{pmatrix}.$$

While $\lambda(\alpha) = 1$, the above equation coincides with Grzegorzewski's formula [12, Equations (29)-(32)]. Note that, the extended trapezoidal approximation $P_{\mathbb{T}}(\tilde{A})$ may be not in $\tilde{\mathbb{F}}$, refer to [4, 5, 20]. This is because that $P_{\mathbb{T}}(\tilde{A})$ may happen $t_2 > t_3$.

Lemma 6. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_{\mathbb{T}}(\tilde{A}) = t_1\tilde{E}_1 + t_2\tilde{E}_2 + t_3\tilde{E}_3 + t_4\tilde{E}_4,$$

where t_i, $1 \le i \le 4$, are computed by Eq.(13). Then,

$$t_1 \le t_2 \quad and \quad t_3 \le t_4.$$

Proof. Omitted.

From Eq.(8), we find

$$P_{\tilde{\mathbb{T}}}(\tilde{A}) = P_{\tilde{\mathbb{T}}}(P_{\mathbb{T}}(\tilde{A})). \tag{14}$$

Hence, if $P_{\mathbb{T}}(\tilde{A})$ is in $\tilde{\mathbb{F}}$ (by applying Lemma 6, it is equivalent to $t_2 \le t_3$), we obtain

$$P_{\tilde{\mathbb{T}}}(\tilde{A}) = P_{\mathbb{T}}(\tilde{A}).$$

Otherwise, we have $t_2 > t_3$. Consequently, Eq.(14) implies that the trapezoidal approximation $P_{\tilde{\mathbb{T}}}(\tilde{A})$ will be restricted to triangular fuzzy numbers $\tilde{\Delta}$. This leads to

$$P_{\tilde{\mathbb{T}}}(\tilde{A}) = P_{\tilde{\Delta}}(\tilde{A}).$$

By applying Theorem 4, we prove the following theorem which is a generalization of [21, Theorem 4.4].

Theorem 7. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_{\mathbb{T}}(\tilde{A}) = t_1\tilde{E}_1 + t_2\tilde{E}_2 + t_3\tilde{E}_3 + t_4\tilde{E}_4,$$

where t_i, $1 \le i \le 4$, are computed by Eq.(13). If $t_2 \le t_3$, then the trapezoidal approximation of $\tilde{A}$ is

$$P_{\tilde{\mathbb{T}}}(\tilde{A}) = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha].$$

Otherwise, by Eq.(9) compute

$$P_{\Delta}(\tilde{A}) = r_1\tilde{E}_1 + r_2(\tilde{E}_2 + \tilde{E}_3) + r_4\tilde{E}_4.$$

Then, the trapezoidal approximation $P_{\tilde{\mathbb{T}}}(\tilde{A})$ can be determined in the following cases:

1. *If $r_1 \le r_2 \le r_4$, then*
$$P_{\tilde{\mathbb{T}}}(\tilde{A}) = [r_1 + (r_2 - r_1)\alpha, r_4 - (r_4 - r_2)\alpha].$$
2. *If $r_2 < r_1$, then $P_{\tilde{\mathbb{T}}}(\tilde{A}) = [r'_1, r'_4 - (r'_4 - r'_1)\alpha]$, where r'_1 and r'_4 are computed by Eq.(10).*
3. *If $r_2 > r_4$, then $P_{\tilde{\mathbb{T}}}(\tilde{A}) = [r'_1 + (r'_4 - r'_1)\alpha, r'_4]$, where r'_1 and r'_4 are computed by Eq.(11).*

Proof. Omitted.

In 1998, Delgado et al. [7] proposed a symmetric trapezoidal approximation of $\tilde{A}$ under the Euclidean distance between the respective 1/2-levels. In the following, we turn to study the symmetric trapezoidal approximation of $\tilde{A}$ under a weighted L^2-distance. Recall that, an extended trapezoidal fuzzy number $t_1\tilde{E}_1 + t_2\tilde{E}_2 + t_3\tilde{E}_3 + t_4\tilde{E}_4$ is symmetric trapezoidal iff

$$t_2 - t_1 = t_4 - t_3 \ge 0 \quad \text{and} \quad t_2 \le t_3.$$

Let $d = t_2 - t_1$. By substituting $t_2 = t_1 + d$ and $t_3 = t_4 - d$, we obtain

$$t_1\tilde{E}_1 + t_2\tilde{E}_2 + t_3\tilde{E}_3 + t_4\tilde{E}_4 = t_1[1,0] + d[\alpha, -\alpha] + t_4[0,1].$$

Let $\mathbb{T}_s := \text{Span}\,\{[1,0], [\alpha, -\alpha], [0,1]\}$. Suppose that the best approximation of $\tilde{A}$ from $\mathbb{T}_s$ is

$$P_{\mathbb{T}_s}(\tilde{A}) := t_1[1,0] + d[\alpha, -\alpha] + t_4[0,1].$$

By applying Eq.(4), we can solve

$$\begin{pmatrix} t_1 \\ d \\ t_4 \end{pmatrix} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 \\ \lambda_1 & 2\lambda_2 & -\lambda_1 \\ 0 & -\lambda_1 & \lambda_0 \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, [1,0] \rangle_\lambda \\ \langle \tilde{A}, [\alpha, -\alpha] \rangle_\lambda \\ \langle \tilde{A}, [0,1] \rangle_\lambda \end{pmatrix}$$
$$= \frac{1}{\delta} \begin{pmatrix} 2\lambda_0\lambda_2 - \lambda_1^2 & -\lambda_0\lambda_1 & -\lambda_1^2 \\ -\lambda_0\lambda_1 & \lambda_0^2 & \lambda_0\lambda_1 \\ -\lambda_1^2 & \lambda_0\lambda_1 & 2\lambda_0\lambda_2 - \lambda_1^2 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 - U_1 \\ U_0 \end{pmatrix} \tag{15}$$

where $\lambda_i := \int_0^1 \alpha^i \lambda(\alpha) d\alpha$, $i = 0, 1, 2$, and

$$\delta := 2\lambda_0(\lambda_0\lambda_2 - \lambda_1^2) > 0.$$

Lemma 8. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_{\mathbb{T}_s}(\tilde{A}) = t_1[1,0] + d[\alpha, -\alpha] + t_4[0,1],$$

where t_1, d, and t_4 are computed by Eq.(15). Then, $d \ge 0$.

ISBN: 978-989-95079-6-8

Proof. Omitted.

If $P_{\mathbb{T}_s}(\tilde{A}) \in \tilde{\mathbb{F}}$ (by applying Lemma 8, it is equivalent to $t_1 + d \leq t_4 - d$), then it equals the symmetric trapezoidal approximation $P_{\tilde{\mathbb{T}}_s}$. Otherwise, we will have $d > \frac{1}{2}(t_4 - t_1)$. This shows that, in the case the symmetric trapezoidal approximation

$$P_{\tilde{\mathbb{T}}_s}(\tilde{A}) = t_1'[1,0] + d'[\alpha, -\alpha] + t_4'[0,1]. \qquad (16)$$

will satisfy $d' = \frac{1}{2}(t_4' - t_1')$. Substituting into Eq.(16) by $d' = \frac{1}{2}(t_4' - t_1')$, we get

$$P_{\tilde{\mathbb{T}}_s}(\tilde{A}) := t_1'[1 - \frac{1}{2}\alpha, \frac{1}{2}\alpha] + t_4'[\frac{1}{2}\alpha, 1 - \frac{1}{2}\alpha].$$

Since $\Delta_s = \text{Span}\,\{[1 - \frac{1}{2}\alpha, \frac{1}{2}\alpha], [\frac{1}{2}\alpha, 1 - \frac{1}{2}\alpha]\}$, we get

$$P_{\tilde{\mathbb{T}}_s}(\tilde{A}) \in \Delta_s \cap \tilde{\mathbb{F}} = \tilde{\Delta}_s.$$

This implies $P_{\tilde{\mathbb{T}}_s}(\tilde{A})$ is equal to the symmetric triangular approximation $P_{\tilde{\Delta}_s}(\tilde{A})$. Consequently, by applying Theorem 5 we obtain the following theorem.

Theorem 9. *Let $\tilde{A}$ be a fuzzy number, and let*

$$P_{\mathbb{T}_s}(\tilde{A}) = t_1[1,0] + d[\alpha, -\alpha] + t_4[0,1],$$

where t_1, d, and t_4 are computed by Eq.(15). If $d \leq \frac{1}{2}(t_4 - t_1)$, then the symmetric trapezoidal approximation of $\tilde{A}$ is

$$P_{\tilde{\mathbb{T}}_s}(\tilde{A}) = [t_1 + d\alpha, t_4 - d\alpha].$$

Otherwise, it is

$$P_{\tilde{\mathbb{T}}_s}(\tilde{A}) = [r_1 + \frac{r_4 - r_1}{2}\alpha, r_4 - \frac{r_4 - r_1}{2}\alpha],$$

where r_1 and r_4 are computed by Eq.(12).

7 Conclusions

Recently, many scholars studied on computation of interval, triangular, and trapezoidal approximations approximations of fuzzy numbers by applying Langrange multiplier method or Karush-Kuhn-Tucker theorem. In the present paper, we propose a new method for computing these approximations under a weighted distance by applying function approximation theory on Hilbert space.We embed fuzzy numbers into the Hilbert space $L^2_\lambda[0,1] \times L^2_\lambda[0,1]$. Then, by introducing extended trapezoidal fuzzy numbers and applying the reduction principle (Theorem 1), it suffices to compute the best approximations of an extended trapezoidal fuzzy number. Hence, we can easily determine these approximations by choosing a suitable basis. In fact, the weighted distance can be generalized to more general form, as follows

$$\mathrm{d}_\lambda(\tilde{A}, \tilde{B}) = \int_0^1 |A_L - B_L|^2 d\mu_1 + \int_0^1 |A_U - B_U|^2 d\mu_2,$$

where μ_1 and μ_2 are arbitrary positive measures on [0,1].

Acknowledgment

The author is very grateful to Prof. P. Grzegorzewski and Prof. L. Stefanini for their invitation and suggestions, which have been very helpful in improving the presentation of the paper. The author also wishes to thank my advisor Prof. Chen-Lian Chuang for his assistance. This work was supported by National Science Council of Taiwan.

References

[1] S. Abbasbandy and B. Asady, The nearest trapezoidal fuzzy number to a fuzzy quantity, Applied Mathematics and Computation 156(2004) 381-386.

[2] S. Abbasbandy and M. Amirfakhrian, The nearest trapezoidal form of a generalized left right fuzzy number, International Journal of Approximate Reasoning 43(2006) 166-178.

[3] S. Abbasbandy and M. Amirfakhrian, The nearest approximation of a fuzzy quantity in parametric form, Applied Mathematics and Computation 172(2006) 624-632.

[4] T. Allahviranloo and M.A. Firozja, Note on "Trapezoidal approximation of fuzzy numbers", Fuzzy Sets and Systems 158(2007) 747-754.

[5] A. Ban, Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval, Fuzzy Sets and Systems, 159(2008) 1327-1344.

[6] S. Chanas, On the interval approximation of a fuzzy number, Fuzzy Sets and Systems 122(2001) 353-356.

[7] M. Delgado, M.A. Vila and W. Voxman, On a canonical representation of fuzzy number, Fuzzy Sets and Systems 93(1998) 125-135.

[8] F. Deutsch, Best approximation in inner product space, Springer, New York, 2001.

[9] D. Dubois and H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems 24(1987) 279-300.

[10] P. Grzegorzewski, Metrics and orders in space of fuzzy numbers, Fuzzy Sets and Systems 97(1998) 83-94.

[11] P. Grzegorzewski, Nearest interval approximation of a fuzzy number, Fuzzy Sets and Systems 130(2002) 321-330.

[12] P. Grzegorzewski and E. Mrówka, Trapezoidal approximations of fuzzy numbers, Fuzzy Sets and Systems 153(2005) 115-135.

[13] P. Grzegorzewski and E. Mrówka, Trapezoidal approximations of fuzzy numbers-revisited, Fuzzy Sets and Systems 158(2007) 757-768.

[14] P. Grzegorzewski and E. Mrówka, Trapezoidal approximations of fuzzy numbers preserving the expected interval - algorithms and properties, Fuzzy Sets and Systems, 159(2008) 1354-1364.

[15] R. Hassine, F. Karray, A.M. Alimi and M. Selmi, Approximation properties of piece-wise parabolic functions fuzzy logic systems, Fuzzy Sets and Systems 157(2006) 501-515.

[16] S. Heilpern, The expected value of a fuzzy number, Fuzzy Sets and Systems 47(1992) 81-86.

[17] M. Hukuhara, Integration des applications measurables dont la valeur est un compact convexe, Funkcialaj Ekvacioj 10(1967) 205-223.

[18] M. Ma, A. Kandel and M. Friedman, A new approach for defuzzification, Fuzzy Sets and Systems 111(2000) 351-356.

[19] E. N. Nasibov and S. Peker, On the nearest parametric approximation of a fuzzy number, Fuzzy Sets and Systems, 159(2008) 1365-1375.

[20] C.T. Yeh, A note on trapezoidal approximations of fuzzy numbers, Fuzzy Sets and Systems 158(2007) 747-754.

[21] C.T. Yeh, On improving trapezoidal and triangular approximations of fuzzy numbers, International Journal of Approximate Reasoning, 48/1 (2008) 297-313.

[22] C.T. Yeh, Trapezoidal and triangular approximations preserving the expected interval, Fuzzy Sets and Systems, 159(2008) 1345-1353.

[23] W. Zeng and H. Li, Weighted triangular approximation of fuzzy numbers, International Journal of Approximate Reasoning, 46/1 (2007) 137-150.

ISBN: 978-989-95079-6-8