A Monge algorithm for computing the Choquet integral on set systems

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Abstract— In works dealing with capacities (fuzzy measures) and the Choquet integral on finite spaces, it is usually considered that all subsets of the universe are measureable. Hence, all functions are integrable in the sense of Choquet. We consider the situation where some subsets are not measurable (not feasible), so that there are non-integrable functions. Since this is a severe limitation in applications, we study how to extend the Choquet integral to any function. Our results mainly deal with the case where the set of feasible subsets is a distributive lattice.

Keywords— Choquet integral, fuzzy measure, capacity, Monge algorithm, set system, distributive lattice

1 Introduction

The Choquet integral [1] has become a widely used tool in decision making and its properties are well known. With general spaces, the issue of measurability is closely related to integration, that is, a function is integrable if all subsets involved in the integration are measurable. In the case of the Choquet integral, these sets are the level sets of the integrand. Although these facts are standard when considering infinite spaces, most of the time it is considered that for finite spaces, all subsets are measurable. However, there are many situations where this assumption cannot be always true. For example, if the space N is a set of players in a game, some coalitions may not be feasible. If N is a set of states of nature, some event may be impossible to realize, if it is a set of criteria, some combinations of criteria may not correspond to conceivable objects, etc.

In cooperative game theory, already a great deal of research has been done concerning games with nonfeasible coalitions, making various assumptions on the structure of feasible coalitions (distributive lattices, as in Faigle and Kern [2], and in Grabisch and Lange [3], convex geometries as in Bilbao [4], antimatroids [5], regular set systems [6], etc.).

Computing the Choquet integral over these sets systems can be done simply by taking the usual definition. However, there will be many nonintegrable functions, for which the Choquet integral remains undefined. This could be a serious limitation in applications related to the above fields. To the best of our knowledge, the question of how to define the Choquet integral for any function on these sets systems has not been addressed. This is precisely what we aim for in this paper, supposing most of the time that the set system is a distributive lattice. We introduce a Monge algorithm, whose output coincides with the Choquet integral for every integrable function. We will show that the output of this algorithm, called the Monge-Choquet integral, is the smallest extension of the Choquet integral. Dually, we introduce the greatest extension and its associated dual Monge algorithm.

2 Preliminaries

In the whole paper, $N := \{1, 2, \ldots, n\}$.

2.1 Capacities and the Choquet integral

A *set system* $\mathcal{F}$ is any collection of sets in 2^N containing the empty set and N. For any set $A \subseteq N$, we define $\mathcal{F}(A) := \{F \in \mathcal{F} \mid F \subseteq A\}$. A *game* on $\mathcal{F}$ is any function $v : \mathcal{F} \to \mathbb{R}$ such that $v(\emptyset) = 0$. A game is *monotone* if for any $A, B \in \mathcal{F}$ such that $A \subseteq B$, we have $v(A) \leq v(B)$. A monotone game is usually called a *capacity* [1] or *fuzzy measure* [7].

Consider a function $f : N \to \mathbb{R}_+$, denoting $f(i)$ by f_i. We say that f is *measurable* if the family of sets $\{j \in N \mid f(j) \geq f(i)\}, i = 1, \ldots, n$ belongs to $\mathcal{F}$.

Definition 1. Let v be a game on $\mathcal{F}$, and $f : N \to \mathbb{R}_+$ be an measurable function. The *Choquet integral* of f w.r.t. v is defined by:

$$\int f\, dv := \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)}) v(\{\sigma(i), \ldots, \sigma(n)\}) \quad (1)$$

with $f_{\sigma(0)} := 0$, $f_{\sigma(1)} \leq \cdots \leq f_{\sigma(n)}$.

For any $A \subseteq N$, let 1_A denotes the characteristic vector of A. A fundamental property of the Choquet integral is that

$$\int 1_A\, dv = v(A), \quad (2)$$

for any A in $\mathcal{F}$. We give some fundamental results on the Choquet integral, useful for the sequel.

Proposition 1. The Choquet integral is monotonic vs. the game, i.e., if $v \leq v'$ pointwise, then $\int f\, dv \leq \int f\, dv'$ for any measurable f.

Two functions $f, f' \in \mathbb{R}^n_+$ are *comonotonic* if there is no $i, j \in N$ such that $f_i > f_j$ and $f'_i < f'_j$. A functional $I : \mathbb{R}^n_+ \to \mathbb{R}$ is *comonotonic additive* if for any two comonotonic functions $f, f' \in \mathbb{R}^n_+$, we have $I(f + f') = I(f) + I(f')$.

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Proposition 2. (Characterization theorem of Schmeidler, established for continuous spaces [8]) Let $I : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ be a functional. Then I is the Choquet integral w.r.t. a capacity v on 2^N if and only if I is nondecreasing, comonotonic additive, and $I(0) = 0$. Then v is uniquely determined by (2).

Consider the hypercube $[0,1]^n$. The *canonical simplices* of $[0,1]^n$ are those induced by any permutation on N as follows:

$$[0,1]^n_\sigma := \{x \in [0,1]^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\}.$$

Proposition 3. (Interpolation theorem, see Singer [9], and [10, Ch. 5]) The unique functional $I : [0,1]^n \rightarrow [0,1]$, whose value is fixed on vertices of $[0,1]^n$, being linear in each canonical simplex, continuous, and vanishing at 0 is the Choquet integral w.r.t. a game v on 2^N, with $v(A) = I(1_A)$, for all $A \subseteq N$.

2.2 Posets, lattices, and set systems

(see, e.g., [11]) A *poset* is any set P endowed with a binary relation $\leq$, being reflexive, antisymmetric and transitive (partial order). A *lattice* $(L, \leq)$ is a poset such that for any $x, y \in L$ their least upper bound, denoted by $x \vee y$, and greatest lower bound $x \wedge y$ always exist. When L is finite, there always exist a greatest element and a least element in L, called the *top* and the *bottom* element.

Let $(P, \leq)$ be any poset. $Q \subseteq P$ is a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. The set of all downsets of P is denoted by $\mathcal{O}(P)$. Observe that if Q, Q' are downsets of P, then so are $Q \cup Q'$ and $Q \cap Q'$. For any $x \in P$, we denote by $\downarrow x$ the downset generated by x (principal ideal of x), i.e.:

$$\downarrow x := \{y \in P \mid y \leq x\}.$$

More generally, for any subset Q of P, one can compute $\downarrow Q$, the *downset generated by* Q, defined by $\downarrow Q := \bigcup_{x \in Q} \downarrow x$. It is indeed a downset since downsets are closed under union.

For any two elements $x, y \in P$, x *is covered by* y or y *covers* x (denoted by $x \prec y$ or $y \succ x$) if $x < y$ and there is no z such that $x < z < y$. A sequence of elements such that $x \leq y_1 \leq y_2 \leq \cdots \leq z$ is called a *chain* from x to z. If in addition $x \prec y_1 \prec y_2 \cdots \prec z$, the chain is *maximal*.

A lattice is *distributive* if $\vee, \wedge$ obey distributivity. Any poset $(P, \leq)$ generates a distributive lattice of subsets of P ordered by inclusion, which is $\mathcal{O}(P)$ (and reciprocally). Its bottom and top elements are $\emptyset$ and P, respectively. In the distributive lattice $\mathcal{O}(P)$, any maximal chain C from bottom to top has length $|P|$, and corresponds to a permutation σ on P, such that $C = \{\emptyset, \{\sigma(n)\}, \{\sigma(n), \sigma(n-1)\}, \ldots, P\}$.

Let us consider as in Section 2.1 the set N of n elements. By defining a partial order $\leq$ on N, $(N, \leq)$ is a poset and $\mathcal{O}(N)$ is a distributive lattice with $\emptyset, N$ as bottom and top elements, hence it is a set system, having all its maximal chains from $\emptyset$ to N of length n. Conversely, a set system whose all maximal chains from $\emptyset$ to N are of length n is not necessarily a distributive lattice. It is called a *regular set system* [6]. An *antimatroid* $\mathcal{F}$ is a regular set system which is closed under union. A *convex geometry* is a regular set system closed under intersection.

2.3 The Monge algorithm

The original idea of the Monge algorithm goes back to [12]. Monge studied a geometric transportation problem in which a set of locations $s_1, \ldots, s_n$ of mass points has to be matched optimally (in the sense of minimizing the total cost) with another set of locations $t_1, \ldots, t_n$, and proved that optimality was reached if the transportation lines do not cross. This geometric fact can be expressed as follows: if the costs c_{ij} of matching objects s_i with t_j have the "uncrossing" property:

$$c_{ij} + c_{k\ell} \geq c_{\max(i,k),\max(j,\ell)} + c_{\min(i,k),\min(j,\ell)}$$

then the optimal matching is $(s_1, t_1), \ldots, (s_n, t_n)$. This is also called the "north-west corner rule". Translated into the language of set functions, the uncrossing property is in fact submodularity:

$$v(A) + v(B) \geq v(A \cup B) + v(A \cap B).$$

The following algorithm, which we call "Monge algorithm", is based on the previous ideas (see also the greedy algorithm of Lovász for maximizing a linear function over the core of a submodular game [13]). Let $A = [a_{ij}] \in \{0,1\}^{m \times n}$ be a $(0,1)$-matrix with m rows $A_1, \ldots, A_m$ and n columns $A^1, \ldots, A^n$. We assume throughout the paper that A does not contain the null row. Let us put $N = \{1, \ldots, n\}$ the set of column indices. Hence, any row of the matrix can be interpreted as the characteristic vector of a nonempty subset of N.

Consider an (input) vector $f \in \mathbb{R}^n_+$. The Monge algorithm produces an output line-vector $y \in \mathbb{R}^m_+$ as follows.

MONGE ALGORITHM

- (M0) Initialization. Put $f' = f$, $A' = [a'_{ij}] = A$, $N' = N$, $y = 0 \in \mathbb{R}^m$.
- (M1) If the current matrix A' is empty, STOP.
- (M2) Let s be the smallest row index in A'.
- (M3) Find the smallest value α of f'_j for all $j \in N'$ such that $a'_{sj} = 1$. Put t the smallest such j.
- (M4) Updating: $y_s \leftarrow \alpha$, $f'_j \leftarrow f'_j - \alpha$ for all $j \in N'$ such that $a'_{sj} = 1$, $N' \leftarrow N' \setminus t$, suppress all rows A'_i in A' such that $a'_{it} = 1$. Go to (M1).

3 The Monge algorithm and the Choquet integral

Consider a set system $\mathcal{F}$ on N. It can be encoded by a $(0,1)$-matrix A as given in Section 2.3. The rows of A order the subsets of $\mathcal{F}$ in a particular way. An interesting case arises when this order is a linear extension of the decreasing inclusion order, that is, for any subsets S, T such that $S \subseteq T$, the corresponding rows A_{i_S}, A_{i_T} are such that $i_S \geq i_T$. We say that in this case the row order *is compatible with decreasing inclusion*. A row order compatible with increasing inclusion can be defined as well.

Consider next a game v on $\mathcal{F}$. It can be encoded as a vector in $\mathbb{R}^m$, with $m = |\mathcal{F}| - 1$. The next result shows that the Choquet integral can be computed by yv, where y is the output of the Monge algorithm, provided rows of A are ordered according to the (decreasing) inclusion order.

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Proposition 4. Let $\mathcal{F}$ be a set system, and an measurable function $f \in [0,1]^N$. Let v be a game on $\mathcal{F}$.

On the other hand, consider a (0,1)-matrix A encoding the set system $\mathcal{F}$ (except the empty set), such that the row order is compatible with decreasing inclusion. Then the output of the Monge algorithm for f is a vector y, such that yv is the Choquet integral defined in (1).

A question is whether the condition on row ordering is necessary to get this result. The answer is given in the next proposition.

Proposition 5. Assume A is a (0-1)-matrix with no two identical rows. For each nonempty subset S corresponding to a row of A, say A_{i_S}, the output of the algorithm is $y = 1_{i_S}$ (i.e., $y_i = 0$ for all i except for $i = i_S$) when the input is $f = 1_S$, if and only if the row order is compatible with decreasing inclusion.

In other words, for any set system $\mathcal{F}$ and any game v on $\mathcal{F}$, for $f = 1_S$, $S \in \mathcal{F}$, we have $yv = v(S)$, which is by (2) the output of the Choquet integral. This proves the necessity of the condition on the row order in Proposition 4.

4 Extended Choquet integrals for distributive lattices

For any set system $\mathcal{F}$, we define the *Monge-Choquet integral* as the output of the Monge algorithm for any nonnegative input vector. In this section, we restrict to the case of distributive lattices induced by a poset on N, denoted by $(N, \leq)$. We know from Section 3 that the Monge-Choquet integral coincides with the usual Choquet integral for measurable input vectors, provided the rows of A are arranged in an order extending the decreasing inclusion order (otherwise stated, this assumption holds from now on). Hence, the Monge-Choquet integral is an extension of the Choquet integral.

4.1 Computation of the Monge-Choquet integral

We show that the Monge-Choquet integral can be computed in a much simpler way, independent of the order of rows of A, provided this order is compatible with decreasing inclusion. Take f any input vector in $\mathbb{R}^n_+$. We can assume w.l.o.g. (this will be shown later on) that there is a unique permutation σ on N such that $f_{\sigma(1)} < \cdots < f_{\sigma(n)}$.

In the first step of the algorithm, we have $s = 1$, $\alpha = f_{\sigma(1)}$, and $t = \sigma(1)$. In the (initially null) vector y, we put $f_{\sigma(1)}$ at position 1, $\sigma(1)$ is deleted from N, any row in A containing $\sigma(1)$ is deleted, and $f' = f - f_{\sigma(1)}$.

In the second step, s corresponds to the largest subset F in $\mathcal{F}$ not containing $\sigma(1)$, which is uniquely determined by

$$F = \{i \in N \mid i \not\geq \sigma(1)\}$$

where $\geq$ is understood for the poset $(N, \leq)$. $F \in \mathcal{F}$ because F is a downset of $(N, \leq)$, and it is the unique largest such set in $\mathcal{F}$ because $\mathcal{F}$ is closed under union. Since it is the largest one not containing $\sigma(1)$, it is ranked first in A', hence s corresponds to this subset. Next, we look for $\sigma(i) \in F$ such that f' is minimum on F, and we put $t = \sigma(i)$, $\alpha = f_{\sigma(i)} - f_{\sigma(1)}$, and α is put in y at position s. Then $\sigma(i)$ is deleted, and all rows of A' containing $\sigma(i)$. The process continues till A' is empty.

The above shows that the following algorithm computes the Monge-Choquet integral in a simpler way. We recall that for any $A \subseteq N$, $\mathcal{F}(A) := \{F \in \mathcal{F} \mid F \subseteq A\}$.

COMPUTATION OF THE MONGE-CHOQUET INTEGRAL FOR DISTRIBUTIVE LATTICES

(MC0) Initialization: $N' \leftarrow N$.

(MC1) Take F to be the largest subset in $\mathcal{F}(N')$. It is given by $F = \{i \in N \mid i \not\geq j, \forall j \in N \setminus N'\}$. If $F = \emptyset$, goto (MC3).

(MC2) Find the minimum of f over F, call i the smallest index in F for which f is minimum. Put $N' \leftarrow N' \setminus i$. Go to (MC1).

(MC3) Denote by $F_1 = N, F_2, \ldots, F_k$ the sequence of subsets obtained in (MC1), and by $i_1, i_2, \ldots, i_k$ the sequence of indices obtained in (MC2). Then

$$\int f \, dv = \sum_{j=1}^{k} (f_{i_j} - f_{i_{j-1}}) v(F_i), \quad (3)$$

with $f_{i_0} := 0$.

Example 1. Take $n = 5$, and the poset represented on Fig. 1 (left). The corresponding set system $\mathcal{F}$ is on the right. Many

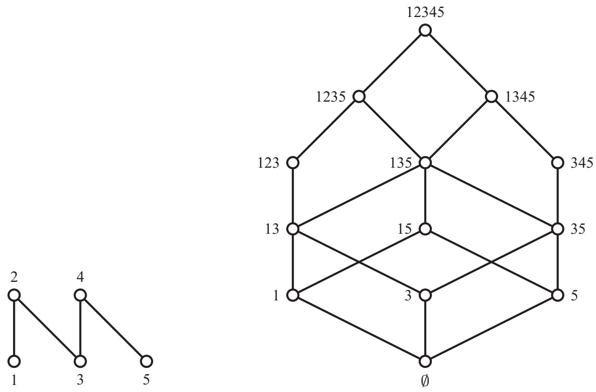

Figure 1: Example of poset with $N = \{1, 2, 3, 4, 5\}$ (left) and the corresponding set system $\mathcal{F}$.

permutations are not compatible with $\mathcal{F}$ (i.e., the induced maximal chain contains sets which are not in $\mathcal{F}$), for example 35241, 12345, 43125, etc. We can anyway compute the corresponding Monge-Choquet integrals:

$$35241: \int f \, dv = f_3 v(N) + (f_5 - f_3) v(15) + (f_1 - f_5) v(1)$$

$$12345: \int f \, dv = f_1 v(N) + (f_3 - f_1) v(345) + (f_5 - f_3) v(5)$$

$$43125: \int f \, dv = f_4 v(N) + (f_3 - f_4) v(1235) + (f_1 - f_3) v(15) + (f_5 - f_1) v(5).$$

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4.2 Properties of the Monge-Choquet integral

The most important property to check is whether the functional is continuous. Obviously it is for a given permutation, but when passing from a permutation to another one, we have to check continuity. If continuity does not hold, we could even say that the integral is not well defined, because a given f may be rearranged in increasing order by several permutations.

Let us take f with permutation σ, such that

$$f_{\sigma(1)} < f_{\sigma(2)} < \cdots < f_{\sigma(i)} = f_{\sigma(i+1)} < \cdots < f_{\sigma(n)}.$$

Hence the permutation σ' obtained by a switch between $\sigma(i)$ and $\sigma(i+1)$ is also rearranging f in nondecreasing order. Observe that in any case, all "compatible" permutations can be obtained by a sufficient number of switches, hence it is enough to study what happens for one switch.

To avoid a heavy notation, and without loss of generality, we may assume that the two following orders are compatible with f:

$$1, 2, \ldots, i, i+1, \ldots, n$$
$$1, 2, \ldots, i+1, i, \ldots, n.$$

The first and second orders will be used by the algorithm when computing the integral of $f' := f + \epsilon 1_{i+1}$ and $f'' := f + \epsilon 1_i$ respectively. Let us check whether the two expressions coincide when ϵ tends towards 0. Clearly, the sequences $F'_1, \ldots, F'_{k'}$ and $F''_1, \ldots, F''_{k''}$ for f' and f'' coincide as long as both $i, i+1$ belong to the sets, and also from the point where both $i, i+1$ disappear. Let us denote respectively by F_0 and F^0 these sets (note that $F^0 = \emptyset$ is possible). Let i_0 be the index of (MC2) when $F = F_0$. Then several cases can occur.

(i) $i_0 < i$ and $i_0 < i+1$ (in the poset). Then F in the next step (MC1) will contain neither i nor $i+1$ for both f', f'' (hence $F = F^0$). Consequently, there will be no difference between the two expressions.

(ii) $i_0 < i$, $i_0 \not< i+1$. Then, the next terms obtained while processing $i, i+1$ will be:

for f': $(f'_{i+1} - f'_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})$
for f'': $(f''_{i+1} - f''_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})$.

(iii) $i_0 < i+1$, $i_0 \not< i$. Similarly,

for f': $(f'_i - f'_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})$
for f'': $(f''_i - f''_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})$.

(iv) $i_0 \not< i$, $i_0 \not< i+1$, and $i < i+1$. Then, the next terms obtained while processing $i, i+1$ will be:

for f': $(f'_i - f'_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})$
for f'': $(f''_{i+1} - f''_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})+$
$(f''_i - f''_{i+1})v(F_0 \cap \{j \in N \mid j \not\geq i_0, j \not\geq i+1\})$.

(v) $i_0 \not< i$, $i_0 \not< i+1$, and $i+1 < i$. Same as above, permuting the roles of f' and f''.

(vi) $i_0 \not< i$, $i_0 \not< i+1$, and $i || i+1$. Then we obtain

for f': $(f'_i - f'_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})+$
$(f'_{i+1} - f'_i)v(F_0 \cap \{j \in N \mid j \not\geq i_0\}, j \not\geq i)$
for f'': $(f''_{i+1} - f''_{i_0})v(F_0 \cap \{j \in N \mid j \not\geq i_0\})+$
$(f''_i - f''_{i+1})v(F_0 \cap \{j \in N \mid j \not\geq i_0, j \not\geq i+1\})$.

In all cases, the expressions become identical when ϵ tends towards 0.

In addition, this proves that in the algorithm, when taking the smallest i when several such i's exist, this choice is unimportant.

From continuity and (3), we easily deduce the following:

Proposition 6. The Monge-Choquet integral is nondecreasing w.r.t. the integrand, positively homogeneous, and is comonotonic additive.

4.3 Smallest and largest extended Choquet integrals

From (3) and the algorithm, it is easy to compute the Monge-Choquet integral for any binary function 1_A, $A \subseteq N$. From Proposition 5, we already know that for any $F \in \mathcal{F}$, $\int 1_F \, dv = v(F)$. Let us define $\hat{v}(A) := \int 1_A \, dv$, for any $A \subseteq N$, which could be seen as an extension of v on 2^N.

Proposition 7. For any $A \subseteq N$, $\hat{v}(A)$ is obtained by

$$\hat{v}(A) = v(F), \quad \text{with } F \text{ the largest subset of } \mathcal{F}(A).$$

Moreover, if v is a capacity, then so is $\hat{v}$, and it is the smallest extension of v over 2^N.

From continuity of the Monge-Choquet integral and Proposition 3, we deduce the following fundamental result.

Proposition 8. For any $f \in \mathbb{R}^n_+$,

$$\int f \, dv = \int f \, d\hat{v}$$

where the left integral is the Monge-Choquet integral, and the right one, the classical Choquet integral.

Hence, the Monge-Choquet integral inherits all properties from the Choquet integral w.r.t. the smallest extension of v over 2^N. In particular, by monotonicity of the integral w.r.t. the capacity (see Proposition 1), we get by Proposition 2:

Corollary 1. Let v be a capacity on $\mathcal{F}$. The Monge-Choquet integral w.r.t. v is the smallest functional $I : \mathbb{R}^n_+ \to \mathbb{R}$ being nondecreasing, comonotonic additive, and such that $I(1_F) = v(F)$ for each F in $\mathcal{F}$.

It is easy to define the largest extension of v over 2^N, denoted by $\check{v}$. It is given by

$$\check{v}(A) := v(\downarrow A)$$

where $\downarrow A$ is the downset generated by A in the poset $(N, \leq)$. Indeed, this is the smallest set in $\mathcal{F}$ containing A. It is possible to obtain $\check{v}$ by an algorithm which is dual to the Monge algorithm.

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DUAL MONGE ALGORITHM. We assume that the row order is compatible with *increasing* inclusion.

(DM0) Init: $A' = [a'_{ij}] = A$, $y = 0 \in \mathbb{R}^m$, $\alpha =$largest value of f, $N' = N$.

(DM1) Choose the smallest index $t \in N'$ such that $f_t = \alpha$. Put $N' \leftarrow N' \setminus t$.

(DM2) Suppress all rows in A' not containing t (i.e., $a'_{s,t} = 0$). Take the smallest row index s in A'.

(DM3) If $N' = \emptyset$ or s is the last row of A' (i.e., it corresponds to N), put $y_s = f_t$ and STOP. Otherwise, look for the highest value α of f_j for all j such that $a'_{s,j} = 0$, put $y_s = f_t - \alpha$, and go to (DM1).

Again, due to the structure of F, a simpler version can be given.

COMPUTATION OF THE DUAL MONGE-CHOQUET INTEGRAL FOR DISTRIBUTIVE LATTICES

(DMC0) Put $N' = \{i_0\}$, where i_0 is the smallest index maximising f.

(DMC1) Take F the smallest set in $\mathcal{F}$ containing N', which is given by $F = \downarrow N'$. If $F = N$, go to (DMC3).

(DMC2) Find the smallest index i maximising f over $N \setminus F$. Put $N' \leftarrow N' \cup \{i\}$. Go to (DMC1).

(DMC3) Denote by $F_1, \dots, F_k$ and $i_0, i_1, \dots, i_{k-1}$ the sequence of sets obtained in (DMC1) and the sequence of indices obtained in (DMC0) and (DMC2) respectively. Then

$$\int f\, dv = \sum_{j=1}^{k} (f_{i_{j-1}} - f_{i_j}) v(F_j) \qquad (4)$$

with $f_{i_k} := 0$.

Example 2. Taking the same poset as in Example 1, we obtain:

$$35241 : \int f\, dv = (f_1 - f_4)v(1) + (f_4 - f_2)v(1345) + f_2 v(N)$$

$$12345 : \int f\, dv = (f_5 - f_4)v(5) + (f_4 - f_2)v(345) + f_2 v(N)$$

$$43125 : \int f\, dv = (f_5 - f_2)v(5) + (f_2 - f_4)v(1235) + f_4 v(N).$$

Proposition 9. The dual Monge-Choquet integral is continuous.

The consequence is that, applying the same reasoning as above, we can deduce the following.

Proposition 10. (i) For any $f \in \mathbb{R}^n_+$,

$$\int^* f\, dv = \int f\, d\check{v}$$

where the left integral is the dual Monge-Choquet integral, and the right one, the classical Choquet integral.

(ii) If v is a capacity on $\mathcal{F}$, then the dual Monge-Choquet integral is the greatest functional $I : \mathbb{R}^n_+ \to \mathbb{R}_+$ being nondecreasing, comonotonic additive, and such that $I(1_F) = v(F)$ for each $F \in \mathcal{F}$.

(iii) For any measurable f, the Monge-Choquet integral and the dual Monge-Choquet integral coincide.

5 The case of regular set systems, antimatroids and convex geometries

5.1 *Regular set systems*

We know from Section 2.1 that a function is measurable if it exists a permutation σ on N ordering f such that the induced sets $\{\sigma(i), \dots, \sigma(n)\}$, $i = 1, \dots, n$, form a maximal chain from $\emptyset$ to N in $\mathcal{F}$, of length n. Hence if $\mathcal{F}$ does not contain maximal chains of length n, there will be no measurable function. This has motivated the definition of regular set systems, whose maximal chains from $\emptyset$ to N are all of length n. In this respect, it would be interesting to know what happens for nonmeasurable functions over regular set systems. It is not difficult to see that, unfortunately, the output of the Monge algorithm depends on the row order, even if it is compatible with inclusion (see example below). The reason is that a regular set system need not be closed under union (nor under intersection), and this property is fundamental for proving results in Section 4. Therefore, the Monge-Choquet integral is not well-defined in this case.

Example 3. Consider $n = 5$ and the regular set system of Fig. 3. Remark that it is not closed under union nor intersection. Hence it is neither an antimatroid nor a convex geometry. Take any function f such that $f_4 < f_3 < f_2 < f_1 < f_5$. Then,

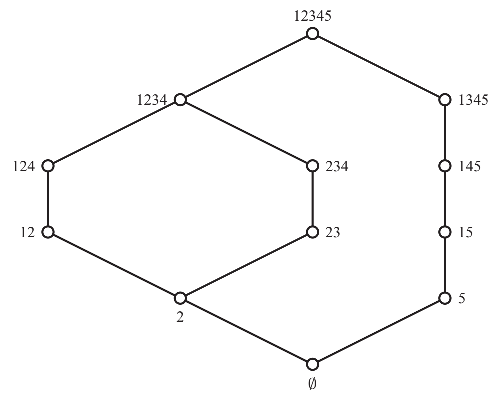

Figure 2: Example of regular set system

in the 1st iteration, $s = 1$, $\alpha = f_4$, $t = 4$, and $f' = f - f_4$. At second iteration, 4 has been discarded, so we have 3 candidates for the next subset: 12, 23, and 15. If 12 is ranked first in the matrix A', we get $\alpha = f_2$, $t = 2$, and 2 is discarded, so only 1 remains. Finally, we get:

$$\int d\, dv = f_4 v(N) + (f_2 - f_4)v(12) + (f_1 - f_2)v(1).$$

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Now, if 23 is ranked first, $\alpha = f_3$, $t = 3$, and we will get:

$$\int f\, dv = f_4 v(N) + (f_3 - f_4)v(23) + (f_2 - f_3)v(2).$$

Hence, the result depends on the ordering.

5.2 Antimatroids and convex geometries

Antimatroids are regular set systems closed under union. This crucial property (as exemplified above) makes most of previous results to hold. More precisely, there will be always a unique largest subset in $\mathcal{F}(N')$ for all $N' \subseteq N$. Consequently, the algorithm of computation of the Monge-Choquet integral and especially Equation (3) given in Section 4.1 remain valid, except for Step (MC1), where F is still the largest subset in $\mathcal{F}(N')$, but now it is no more possible to give it explicitly.

An important question is whether continuity still holds. Using a similar argument, one can prove that continuity still holds in the case of antimatroids. A sketch of the proof goes as follows. Keeping the same notation as in Section 4.2, consider functions f', f'', whose orderings differ only on i and $i+1$. There is a common chain from N to F_0 for both functions f' and f''. From F_0 which contains both i and $i+1$, functions f' and f'' may have different chains, but these chains will necessarily rejoin in F^0 (because there is a unique largest subset contained in $\{i+2, \ldots, n\}$). At F_0, the new term in the integral is $(f'_i - f'_k)v(F_0)$ for f' and $(f''_{i+1} - f''_k)v(F_0)$ for f'' for some k among $1, \ldots, i-1$. At F^0, the new term is $(f'_k - f'_{i+1})v(F^0)$ for f' and $(f''_k - f''_i)v(F^0)$ for f'' for some k in F^0, assuming $i, i+1$ are present in the preceding step (other cases work similarly). Between F_0 and F^0, there is at most one set F on each chain, where the new term will be $(f'_{i+1} - f'_i)v(F)$, and similarly for f''. When ϵ tends towards 0, both integrals coincide.

Consequently, Propositions 6, 7, 8, and Corollary 1 still hold.

On the contrary, it is no more possible to define the largest extension $\breve{v}$ and consequently the dual Monge-Choquet integral, because since an antimatroid is not closed under intersection, it may happen that there is no smallest set of $\mathcal{F}$ containing in a given subset of N.

In a dual way, since convex geometries are closed under intersection but not under union, the dual Monge-Choquet integral exists and possesses all properties given in Section 4.3, while the Monge-Choquet integral is not well-defined.

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