Decomposition of Interval-Valued Fuzzy Morphological Operations by Weak $[\alpha_1, \alpha_2]$-cuts

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Abstract— *One of the extensions of binary morphology to greyscale images is given by the classical fuzzy mathematical morphology. Interval-valued fuzzy mathematical morphology further extends the latter theory by now also allowing uncertainty in the grey values of the image. In this paper, the decomposition of the interval-valued fuzzy morphological operations into their binary counterparts is studied both in a general continuous framework and a discrete framework. It will be shown that some properties that do not hold in the continuous framework, do hold in the discrete framework, which is the framework that is used in practice.*

Keywords— Decomposition, interval-valued fuzzy sets, mathematical morphology.

1 Introduction

The aim of many image processing theories is to extract specific information out of an image. Mathematical morphology is one of those theories. The basic morphological operations dilation, erosion, opening and closing transform a given image into another image by the help of a structuring element. In the original binary morphology [1], both the image and the structuring element needed to be binary images (i.e., black-white). The threshold approach [1] extended the binary morphology by allowing greyscale images. Another extension, the umbra approach [2], even allowed both greyscale images and structuring elements. Some time later, one more greyscale approach was introduced, based on fuzzy set theory [3, 4]. Greyscale images and fuzzy sets can namely both be modelled as a mapping from a universe $\mathcal{U}$ into the unit interval $[0, 1]$. Remark however that fuzzy set theory is only used as a tool here and not to deal with uncertainty. Recently, also extensions of the fuzzy mathematical morphology have arised [5, 6]. The interval-valued fuzzy extension that we consider in this paper, now allows uncertainty regarding the grey values in the image, since a coordinate in the image domain is no longer mapped onto a specific grey value belonging to the unit interval $[0, 1]$, but onto an interval of grey values ($\subseteq [0, 1]$) to which the uncertain grey value is expected to belong. The uncertainty in grey values can have several causes. (i) In practice, we have to deal with the fact that any device rounds off a captured grey value to an element of a finite set of allowed values; (ii) Taking several shots of the same image may result in different grey values for some of the image pixels. This can sometimes happen under identical recording circumstances and is very likely to be the case under changing circumstances (e.g. illumination changes due to clouds passing the sun). The difference in grey value can also be the result of a slight shift in position of the camera or an object in the image between takes. This will especially cause (mostly large) uncertainty at pixels belonging to the edge of an object. (iii) There might also exist uncertainty regarding the grey values in the structuring element that is used. This structuring element can be chosen by the user and sometimes the user might doubt which weight (importance) to give to a certain pixel in the structuring element. In all of the above cases, it might be advisible to work with intervals of likely values instead of one specific value. The interval-valued fuzzy set theory now thus also serves as a model and not only as a tool [6].

In this paper, the relationships between the weak $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy dilation, erosion, opening and closing and the corresponding binary operations are studied. Such decomposition properties are useful (i) to compute interval-valued fuzzy morphological operations by applying several binary operations and combining the obtained results and (ii) to approximate interval-valued fuzzy morphological operations by only considering a finite number of $[\alpha_1, \alpha_2]$-cuts.

The remainder of the paper is organized as follows: the basic notions on interval-valued fuzzy mathematical morphology are introduced in section 2; section 3 studies the decomposition of the interval-valued fuzzy morphological operations by weak $[\alpha_1, \alpha_2]$-cuts in a continuous framework and investigates whether a discrete framework affects those results. Section 4 finally, concludes the paper.

2 Interval-Valued Fuzzy Mathematical Morphology

2.1 *Interval-valued Fuzzy Set Theory*

A fuzzy set [7] F defined over a universe $\mathcal{U}$, maps every element $u \in \mathcal{U}$ onto its membership degree $F(u) \in [0, 1]$ in that set F and allows in this way a gradual transition between belonging to (membership degree 1) and not belonging to (membership degree 0). Interval-valued fuzzy sets [8] extend the classical fuzzy sets by allowing uncertainty concerning the membership degree. They map an element of the universe onto an interval of values instead of one specific membership degree. In other words, they are modelled by mappings from the universe $\mathcal{U}$ into the class of closed intervals $L^I = \{[x_1, x_2] | [x_1, x_2] \subseteq [0, 1]\}$. This means that for an interval-valued fuzzy set G in a universe $\mathcal{U}$, $G(u) = [G_1(u), G_2(u)] \subseteq [0, 1]$, $\forall u \in \mathcal{U}$. In this paper, the class of

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interval-valued sets over the universe $\mathcal{U}$ will be denoted by $\mathcal{IVFS}(\mathcal{U})$. The lower and upper bound of an element x of L^I is denoted by respectively x_1 and x_2, i.e., $x = [x_1, x_2]$. Further, we will restrict the universe $\mathcal{U}$ in the sequel to $\mathbb{R}^n$, corresponding to the coordinates of an n-dimensional image.

Consider now the following partial ordering $\leq_{L^I}$ on L^I:

$$x \leq_{L^I} y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2,\ \forall x, y \in L^I.$$

It can be shown that $(L^I, \leq_{L^I})$ forms a complete lattice [9]. The infimum and supremum of a subset S of L^I are then given by $\inf S = [\inf_{x\in S} x_1, \inf_{x\in S} x_2]$ and $\sup S = [\sup_{x\in S} x_1, \sup_{x\in S} x_2]$ respectively. In the remainder, we will use the notations 0_{L^I} for $\inf L^I = [0, 0]$ and 1_{L^I} for $\sup L^I = [1, 1]$ and we will also use the following orderings on L^I $(\forall x, y \in L^I)$:

$$\begin{aligned} x \ll_{L^I} y &\Leftrightarrow x_1 < y_1 \text{ and } x_2 < y_2, \\ x \geq_{L^I} y &\Leftrightarrow y \leq_{L^I} x, \\ x \gg_{L^I} y &\Leftrightarrow y \ll_{L^I} x. \end{aligned}$$

2.2 Binary and Interval-Valued Fuzzy Mathematical Morphological Operations

In this paper, the decomposition of the interval-valued fuzzy morphological operations into their binary counterparts is studied. Therefore, we first refresh the binary morphological operations.

Definition 1. [1] *Let $A, B \subseteq \mathbb{R}^n$. The binary dilation $D(A, B)$, erosion $E(A, B)$, closing $C(A, B)$ and opening $O(A, B)$ are the sets given by:*

$$\begin{aligned} D(A, B) &= \{y | T_y(B) \cap A \neq \emptyset\}, \\ E(A, B) &= \{y | T_y(B) \subseteq A\}, \\ C(A, B) &= E(D(A, B), -B), \\ O(A, B) &= D(E(A, B), -B), \end{aligned}$$

with $T_y(B) = \{x \in \mathbb{R}^n | x - y \in B\}$ and $-B = \{-b | b \in B\}$.

It is clear that the notions of intersection and inclusion are quite important in the above definitions. Hence, to extend the binary morphological operations to interval-valued fuzzy operators, the underlying Boolean conjunction and implication are extended by conjunctors and implicators on L^I. First, however, we extend the Boolean negation by a negator on L^I.

A *negator* $\mathcal{N}$ on L^I is a decreasing $L^I - L^I$ mapping that coincides with the Boolean negation on $\{0, 1\}$ $(\mathcal{N}(0_{L^I}) = 1_{L^I}$ and $\mathcal{N}(1_{L^I}) = 0_{L^I})$. If $(\forall x \in L^I)(\mathcal{N}(\mathcal{N}(x)) = x)$, then the negator $\mathcal{N}$ is called *involutive*. The standard negator $\mathcal{N}_s$, given by $\mathcal{N}_s([x_1, x_2]) = [1-x_2, 1-x_1]$, for all $x = [x_1, x_2] \in L^I$, is an example of an involutive negator on L^I.

A *conjunctor* $\mathcal{C}$ on L^I is an increasing $(L^I)^2 - L^I$ mapping that coincides with the Boolean conjunction on $\{0, 1\}^2$ $(\mathcal{C}(0_{L^I}, 0_{L^I}) = \mathcal{C}(0_{L^I}, 1_{L^I}) = \mathcal{C}(1_{L^I}, 0_{L^I}) = 0_{L^I}$ and $\mathcal{C}(1_{L^I}, 1_{L^I}) = 1_{L^I})$. If a conjunctor $\mathcal{C}$ satisfies $(\forall x \in L^I)(\mathcal{C}(1_{L^I}, x) = \mathcal{C}(x, 1_{L^I}) = x)$, then it is called a *semi-norm* on L^I. Furthermore, if a semi-norm $\mathcal{C}$ is commutative and associative, then we call it a *t-norm* on L^I. The conjunctor $\mathcal{C}_{\min}$, defined by $\mathcal{C}_{\min}(x, y) = [\min(x_1, y_1), \min(x_2, y_2)]$, for all $(x, y) \in (L^I)^2$, is an example of a t-norm on L^I.

Finally, an *implicator* $\mathcal{I}$ on L^I is a hybrid monotonic $(L^I)^2 - L^I$ mapping (i.e., decreasing in the first argument and increasing in the second argument) that coincides with the Boolean implication on $\{0, 1\}^2$ $(\mathcal{I}(0_{L^I}, 0_{L^I}) = \mathcal{I}(0_{L^I}, 1_{L^I}) = \mathcal{I}(1_{L^I}, 1_{L^I}) = 1_{L^I}$ and $\mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I})$. It can be checked that for every implicator $\mathcal{I}$, the operation $\mathcal{N}_{\mathcal{I}}$ defined by $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0_{L^I}), \forall x \in L^I$, is a negator on L^I. If an implicator $\mathcal{I}$ satisfies $(\forall x \in L^I)(\mathcal{I}(1_{L^I}, x) = x)$, then it is called a *border implicator* on L^I. Furthermore, if a border implicator $\mathcal{I}$ is contrapositive w.r.t. its induced negator, i.e., $(\forall (x, y) \in (L^I)^2)(\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(y), \mathcal{N}_{\mathcal{I}}(x)))$, and if it fulfills the exchange principle, i.e., $(\forall (x, y, z) \in (L^I)^3)(\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)))$, then we call it a *model implicator* on L^I. The implicator $\mathcal{I}_{\max,\mathcal{N}_s}$ given by $\mathcal{I}_{\max,\mathcal{N}_s}(x, y) = [\max(1 - x_2, y_1), \max(1 - x_1, y_2)]$, for all $(x, y) \in (L^I)^2$, is an example of a model implicator on L^I.

Using the above concepts, we can extend the binary morphological operations to the interval-valued fuzzy case.

Definition 2. *Let $\mathcal{C}$ be a conjunctor on L^I, let $\mathcal{I}$ be an implicator on L^I, and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. The interval-valued fuzzy dilation $D^I_{\mathcal{C}}(A, B)$, erosion $E^I_{\mathcal{I}}(A, B)$, closing $C^I_{\mathcal{C},\mathcal{I}}(A, B)$ and opening $O^I_{\mathcal{C},\mathcal{I}}(A, B)$ are the interval-valued fuzzy sets in $\mathbb{R}^n$ defined for all $y \in \mathbb{R}^n$ by:*

$$\begin{aligned} D^I_{\mathcal{C}}(A, B)(y) &= \sup_{x \in T_y(d_B) \cap d_A} \mathcal{C}(B(x - y), A(x)), \\ E^I_{\mathcal{I}}(A, B)(y) &= \inf_{x \in T_y(d_B)} \mathcal{I}(B(x - y), A(x)), \\ C^I_{\mathcal{C},\mathcal{I}}(A, B)(y) &= E^I_{\mathcal{I}}(D^I_{\mathcal{C}}(A, B), -B)(y), \\ O^I_{\mathcal{C},\mathcal{I}}(A, B)(y) &= D^I_{\mathcal{C}}(E^I_{\mathcal{I}}(A, B), -B)(y), \end{aligned}$$

with $d_A = \{x | x \in \mathbb{R}^n$ and $A(x) \neq 0_{L^I}\}$, $d_B = \{x | x \in \mathbb{R}^n$ and $B(x) \neq 0_{L^I}\}$ and $(-B)(x) = B(-x), \forall x \in \mathbb{R}^n$.

Remark that if $y \notin D(d_A, d_B)$, then $D^I_{\mathcal{C}}(A, B)(y) = 0_{L^I}$.

In [11] it is shown that fuzzy mathematical morphology is compatible with binary morphology and if we restrict ourselves to semi-norms and border implicators it is also compatible with greyscale morphology based on the threshold approach. Since the interval-valued fuzzy morphology is compatible with the fuzzy morphology and because we want to preserve also the compatibility with the threshold approach, we will restrict ourselves in the remainder to semi-norms and border implicators on L^I.

2.3 The Discrete Framework

For the practical processing of an image, one has to deal with the technical limitations of a computer. To store an image, the domain is downsampled from the continuous space $\mathbb{R}^n$ to the discrete space $\mathbb{Z}^n$, and the image is represented by a matrix with a given number of rows and columns. Also the possible grey values are sampled. Grey values do not longer belong to the complete continuous unit interval $[0, 1]$, but are downsampled to a finite subchain of it. In the interval-valued fuzzy mathematical morphology, the used intervals of possible grey values are now thus subsets of the finite subchain $L^I_{r,s}$ of L^I, with $L^I_{r,s} = \{[\frac{r-k}{r-1}, \frac{s-l}{s-1}] | k, l \in \mathbb{Z}$ and $1 \leq k \leq r$ and $1 \leq l \leq s\}$ for given integers r and s. We denote the class of all interval-valued fuzzy sets in $\mathbb{Z}^n$ with membership intervals in $L^I_{r,s}$ as $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. Further, remark that for an interval-valued fuzzy set $A \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n), \forall x \in \mathbb{Z}^n$, $A_1(x) \in I_r = \{\frac{r-k}{r-1} | k \in \mathbb{Z}$ and $1 \leq k \leq r\}$ and $A_2(x) \in I_s\{\frac{s-l}{s-1} | l \in \mathbb{Z}$ and $1 \leq l \leq s\}$.

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The concepts of negators, conjunctors and implicators on the chain $L^I_{r,s}$ can be adopted from the previous subsection by replacing L^I by $L^I_{r,s}$.

The definitions of the discrete interval-valued fuzzy dilation and erosion can now be written as follows:

Definition 3. *Let $\mathcal{C}$ be a conjunctor on $L^I_{r,s}$, let $\mathcal{I}$ be an implicator on $L^I_{r,s}$, and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. The discrete interval-valued fuzzy dilation $D^I_{\mathcal{C}}(A,B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ and erosion $E_{\mathcal{I}}(A,B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ are defined by:*

$$\begin{aligned} D^I_{\mathcal{C}}(A,B)(y) &= [\max_{x \in T_y(d_B) \cap d_A} \mathcal{C}(B(x-y), A(x))_1, \\ &\quad \max_{x \in T_y(d_B) \cap d_A} \mathcal{C}(B(x-y), A(x))_2], \\ E^I_{\mathcal{I}}(A,B)(y) &= [\min_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x))_1, \\ &\quad \min_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x))_2]. \end{aligned}$$

3 Decomposition of Interval-valued Fuzzy Morphological Operations

3.1 Weak $[\alpha_1, \alpha_2]$-cuts

We first introduce the different weak $[\alpha_1, \alpha_2]$-cuts of an interval-valued fuzzy set [10].

Definition 4. *Let $A \in \mathcal{IVFS}(\mathbb{R}^n)$. For $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$, the weak $[\alpha_1, \alpha_2]$-cut $A^{\alpha_2}_{\alpha_1}$ of A is given by:*

$$\begin{aligned} A^{\alpha_2}_{\alpha_1} &= \{x | x \in \mathbb{R}^n,\ A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2\} \\ &= \{x | x \in \mathbb{R}^n \text{ and } A(x) \geq_{L^I} [\alpha_1, \alpha_2]\}. \end{aligned}$$

For $\alpha_1 \in]0,1]$, the weak α_1-subcut A_{α_1} of A is given by:

$$A_{\alpha_1} = \{x | x \in \mathbb{R}^n \text{ and } A_1(x) \geq \alpha_1\}.$$

For $\alpha_2 \in]0,1]$, the weak α_2-supercut A^{α_2} of A is given by:

$$A^{\alpha_2} = \{x | x \in \mathbb{R}^n \text{ and } A_2(x) \geq \alpha_2\}.$$

The cases $[\alpha_1, \alpha_2] = 0_{L^I}$, $\alpha_1 = 0$ and $\alpha_2 = 0$ are excluded for respectively the weak $[\alpha_1, \alpha_2]$-cut, α_1-subcut and α_2-supercut. Since $\{x | x \in \mathbb{R}^n,\ A_1(x) \geq 0 \text{ and } A_2(x) \geq 0\} = \{x | x \in \mathbb{R}^n \text{ and } A_1(x) \geq 0\} = \{x | x \in \mathbb{R}^n \text{ and } A_2(x) \geq 0\} = \mathbb{R}^n$, these cases don't yield new information.

3.2 Decomposition of the Interval-valued Fuzzy Dilation

Lemma 1. *If $\mathcal{C}$ is a semi-norm on L^I, then it holds that $\mathcal{C} \leq \mathcal{C}_{\min}$, i.e.: $(\forall (x,y) \in (L^I)^2)(\mathcal{C}(x,y) \leq_{L^I} \mathcal{C}_{\min}(x,y))$.*

Proof. Let $\mathcal{C}$ be a semi-norm on L^I, then it holds for all $(x,y) \in (L^I)^2$ that on the one hand $\mathcal{C}(x,y) \leq_{L^I} \mathcal{C}(x, 1_{L^I})$ and $\mathcal{C}(x, 1_{L^I}) = x$, and on the other hand $\mathcal{C}(x,y) \leq_{L^I} \mathcal{C}(1_{L^I}, y)$ and $\mathcal{C}(1_{L^I}, y) = y$, from which it can be concluded that $\mathcal{C}(x,y) \leq_{L^I} \mathcal{C}_{\min}(x,y)$. □

3.2.1 Decomposition by weak $[\alpha_1, \alpha_2]$-cuts

There is no relationship between the weak $[\alpha_1, \alpha_2]$-cut $D^I_{\mathcal{C}}(A,B)^{\alpha_2}_{\alpha_1}$ and the binary dilation $D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1})$ that holds in general for an arbitrary semi-norm $\mathcal{C}$.

Example 1. Let $[\alpha_1, \alpha_2] = [1/4, 1]$, $A(x) = [x/2, x]$ for all $x \in [0,1[$, $A(x) = 0_{L^I}$ for all $x \in \mathbb{R} \backslash [0,1[$, $B(x) = 1_{L^I}$ for all $x \in [0,1]$ and $B(x) = 0_{L^I}$ for all $x \in \mathbb{R} \backslash [0,1]$. So $d_A =]0,1[$, $d_B = [0,1]$ and $D(d_A, d_B) =]-1,1[$ and consequently $0 \in D(d_A, d_B)$. Let $\mathcal{C}$ be an arbitrary semi-norm.

It then holds that

$$\begin{aligned} D^I_{\mathcal{C}}(A,B)(0) &= \sup_{x \in T_0(d_B) \cap d_A} \mathcal{C}(B(x), A(x)) = \\ \sup_{x \in]0,1[} \mathcal{C}(1_{L^I}, [x/2, x]) &= \sup_{x \in]0,1[} [x/2, x] = [1/2, 1], \end{aligned}$$

which means that $0 \in D^I_{\mathcal{C}}(A,B)^1_{0.25}$.

On the other hand, however, since $A^1_{0.25} = \emptyset$ also $D(A^1_{0.25}, B^1_{0.25}) = \emptyset$ and thus $0 \notin D(A^1_{0.25}, B^1_{0.25})$. As a consequence $D^I_{\mathcal{C}}(A,B)^1_{0.25} \not\subseteq D(A^1_{0.25}, B^1_{0.25})$.

Neither does it hold in general that $D^I_{\mathcal{C}}(A,B)^{\alpha_2}_{\alpha_1} \supseteq D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1})$. Take for example $[\alpha_1, \alpha_2] = [1/4, 1/2]$, $\mathcal{C}(r,s) = [r_1 \cdot s_1, r_2 \cdot s_2]$ for all $r, s \in L^I$, $A(x) = [0.3, 0.6]$ for all $x \in [0,1]$, $A(x) = 0_{L^I}$ for all $x \in \mathbb{R} \backslash [0,1]$, $B(x) = [0.4, 0.7]$ for all $x \in [0,1]$ and $B(x) = 0_{L^I}$ for all $x \in \mathbb{R} \backslash [0,1]$).

Then on the one hand $0 \in D(A^{0.5}_{0.25}, B^{0.5}_{0.25}) = D(d_A, d_B) = [-1, 1]$.

On the other hand however:

$$\begin{aligned} D^I_{\mathcal{C}}(A,B)(0) &= \sup_{x \in T_0(d_B) \cap d_A} \mathcal{C}(B(x), A(x)) = \\ \sup_{x \in [0,1]} [0.3 \cdot 0.4, 0.6 \cdot 0.7] &= [0.12, 0.42] \not\geq_{L^I} [0.25, 0.5], \end{aligned}$$

or thus $0 \notin D^I_{\mathcal{C}}(A,B)^{0.5}_{0.25}$. As a consequence $D^I_{\mathcal{C}}(A,B)^{0.5}_{0.25} \not\supseteq D(A^{0.5}_{0.25}, B^{0.5}_{0.25})$.

◊

For the semi-norm $\mathcal{C} = \mathcal{C}_{\min}$, we have the following partial result.

Proposition 1. *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$ that:*

$$D^I_{\mathcal{C}_{\min}}(A,B)^{\alpha_2}_{\alpha_1} \supseteq D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1}).$$

Proof. Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$. Then we have:

$$\begin{aligned} & y \in D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1}) \\ \Leftrightarrow\ & T_y(B^{\alpha_2}_{\alpha_1}) \cap A^{\alpha_2}_{\alpha_1} \neq \emptyset \\ \Leftrightarrow\ & (\exists x \in T_y(d_B) \cap d_A)(x \in T_y(B^{\alpha_2}_{\alpha_1}) \text{ and } x \in A^{\alpha_2}_{\alpha_1}) \\ \Leftrightarrow\ & (\exists x \in T_y(d_B) \cap d_A) \\ & (B(x-y) \geq_{L^I} [\alpha_1, \alpha_2] \text{ and } A(x) \geq_{L^I} [\alpha_1, \alpha_2]) \\ \Leftrightarrow\ & (\exists x \in T_y(d_B) \cap d_A)([\min(B_1(x-y), A_1(x)), \\ & \min(B_2(x-y), A_2(x))] \geq_{L^I} [\alpha_1, \alpha_2]) \\ \Leftrightarrow\ & (\exists x \in T_y(d_B) \cap d_A) \\ & (\mathcal{C}_{\min}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2]) \\ \Rightarrow\ & \sup_{x \in T_y(d_B) \cap d_A} \mathcal{C}_{\min}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2] \\ \Leftrightarrow\ & D^I_{\mathcal{C}_{\min}}(A,B)(y) \geq_{L^I} [\alpha_1, \alpha_2] \\ \Leftrightarrow\ & y \in D^I_{\mathcal{C}_{\min}}(A,B)^{\alpha_2}_{\alpha_1}. \end{aligned}$$

□

The reverse inclusion $D^I_{\mathcal{C}_{\min}}(A,B)^{\alpha_2}_{\alpha_1} \subseteq D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1})$ does not hold in general as already illustrated in Example 1.

Remark that the above decomposition property for weak $[\alpha_1, \alpha_2]$-cuts remains valid in the discrete framework.

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3.2.2 Decomposition by weak sub- and supercuts

There is also no relationship between the weak sub- and supercut $D^I_{\mathcal{C}}(A,B)_{\alpha_1}$ and $D^I_{\mathcal{C}}(A,B)^{\alpha_2}$ and the binary dilations $D(A_{\alpha_1},B_{\alpha_1})$ and $D(A^{\alpha_2},B^{\alpha_2})$ that holds in general for an arbitrary semi-norm $\mathcal{C}$. To illustrate this, we can use Example 1 again, where the weak $[\alpha_1,\alpha_2]$-cuts and the weak super- and subcuts of A and B coincide and the results thus remain valid for weak sub- and supercuts.

For the semi-norm $\mathcal{C}=\mathcal{C}_{\min}$, we have the following partial result.

Proposition 2. *Let $A,B\in\mathcal{IVFS}(\mathbb{R}^n)$, then it holds that:*

(i) $(\forall\alpha_1\in]0,1])(D^I_{\mathcal{C}_{\min}}(A,B)_{\alpha_1}\supseteq D(A_{\alpha_1},B_{\alpha_1}))$,
(ii) $(\forall\alpha_2\in]0,1])(D^I_{\mathcal{C}_{\min}}(A,B)^{\alpha_2}\supseteq D(A^{\alpha_2},B^{\alpha_2}))$.

Proof. Analogous to the proof of Proposition 1. □

The reverse inclusion does not hold, as already illustrated in Example 1, where replacing the weak $[\alpha_1,\alpha_2]$-cuts of A and B by the coinciding weak sub- and supercuts doesn't affect the results.

In the discrete framework, not only does Proposition 2 still remain valid, but the result now also holds for arbitrary semi-norms. Further, for $\mathcal{C}_{\min}$ also the reverse inclusion now holds.

Proposition 3. *Let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds that:*
(i) $(\forall\alpha_1\in]0,1]\cap I_r)(D^I_{\mathcal{C}_{\min}}(A,B)_{\alpha_1}=D(A_{\alpha_1},B_{\alpha_1}))$,
(ii) $(\forall\alpha_2\in]0,1]\cap I_s)(D^I_{\mathcal{C}_{\min}}(A,B)^{\alpha_2}=D(A^{\alpha_2},B^{\alpha_2}))$.

Proof. Analogous to the proof of Proposition 2, where now in the discrete case also

$$(\exists x\in T_y(d_B)\cap d_A)(\mathcal{C}_{\min}(B(x-y),A(x))_1\geq\alpha_1)$$
$$\Updownarrow$$
$$\sup_{x\in T_y(d_B)\cap d_A}\mathcal{C}_{\min}(B(x-y),A(x))_1\geq\alpha_1.$$

□

Proposition 4. *Let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds that:*
(i) $(\forall\alpha_1\in]0,1]\cap I_r)(D^I_{\mathcal{C}}(A,B)_{\alpha_1}\subseteq D(A_{\alpha_1},B_{\alpha_1}))$,
(ii) $(\forall\alpha_2\in]0,1]\cap I_s)(D^I_{\mathcal{C}}(A,B)^{\alpha_2}\subseteq D(A^{\alpha_2},B^{\alpha_2}))$.

Proof. Analogous to the proof of Proposition 3, but for an arbitrary semi-norm $\mathcal{C}$, so that

$$D^I_{\mathcal{C}_{\min}}(A,B)(y)_1\geq\alpha_1$$
$$\Uparrow$$
$$D^I_{\mathcal{C}}(A,B)(y)_1\geq\alpha_1$$
$$\Updownarrow$$
$$y\in D^I_{\mathcal{C}}(A,B)_{\alpha_1}.$$

□

3.3 Decomposition of the Interval-valued Fuzzy Erosion

Based on its induced negator $\mathcal{N}_{\mathcal{I}}$, a border implicator $\mathcal{I}$ can be classified as an upper or a lower border implicator as follows. A border implicator on L^I is called an upper border implicator (respectively lower border implicator) if $\mathcal{N}_{\mathcal{I}}\geq\mathcal{N}_s$ (respectively $\mathcal{N}_{\mathcal{I}}\leq\mathcal{N}_s$).

Lemma 2. *If $\mathcal{I}$ is an upper border implicator on L^I, then it holds that $\mathcal{I}\geq\mathcal{I}_{\max,\mathcal{N}_s}$, i.e.: $(\forall(x,y)\in(L^I)^2)(\mathcal{I}(x,y)\geq_{L^I}\mathcal{I}_{\max,\mathcal{N}_s}(x,y)$.*

Proof. Let $\mathcal{I}$ be an upper border implicator on L^I. For all $(x,y)\in(L^I)^2$ it holds that on the one hand $\mathcal{I}(x,y)\geq_{L^I}\mathcal{I}(1_{L^I},y)$ and $\mathcal{I}(1_{L^I},y)=y$, and on the other hand $\mathcal{I}(x,y)\geq_{L^I}\mathcal{I}(x,0_{L^I})$ and $\mathcal{I}(x,0_{L^I})\geq_{L^I}\mathcal{N}_s(x)$, from which it follows that $\mathcal{I}(x,y)\geq_{L^I}\mathcal{I}_{\max,\mathcal{N}_s}(x,y)$. □

3.3.1 Decomposition by weak $[\alpha_1,\alpha_2]$-cuts

As illustrated below, there is no relationship between the weak $[\alpha_1,\alpha_2]$-cut $E^I_{\mathcal{I}}(A,B)^{\alpha_2}_{\alpha_1}$ and the binary erosion $E(A^{\alpha_2}_{\alpha_1},B^{\overline{1-\alpha_1}}_{\overline{1-\alpha_2}})$ that holds in general for an arbitrary upper border implicator $\mathcal{I}$.

Example 2. Let $[\alpha_1,\alpha_2]=[0.4,0.6]$, $A(x)=[0.3,0.5]$ for all $x\in[0,1]$, $B(x)=[0.5,0.7]$ for all $x\in[0,1]$ and $A(x)=B(x)=0_{L^I}$ for all $x\in\mathbb{R}\backslash[0,1]$. Let $\mathcal{I}$ be the upper border implicator given by $\mathcal{I}_L(x,y)=[\min(1,1-x_2+y_1),\min(1,1-x_1+y_2)]$, $\forall(x,y)\in(L^I)^2$, which is a generalisation of the Łukasiewicz implicator on $[0,1]$.

On the one hand, we have that

$$E^I_{\mathcal{I}_L}(A,B)(0)=\inf_{x\in T_0(d_B)}\mathcal{I}_L(B(x),A(x))=$$
$$\inf_{x\in[0,1]}\mathcal{I}_L([0.5,0.7],[0.3,0.5])=[0.6,1],$$

or thus $0\in E^I_{\mathcal{I}_L}(A,B)^{0.6}_{0.4}$.

On the other hand, $E(A^{0.6}_{0.4},B^{\overline{0.6}}_{\overline{0.4}})=E(\emptyset,[0,1])=\emptyset$ and thus $0\notin E(A^{0.6}_{0.4},B^{\overline{0.6}}_{\overline{0.4}})$, which implies that $E^I_{\mathcal{I}_L}(A,B)^{0.6}_{0.4}\not\subseteq E(A^{0.6}_{0.4},B^{\overline{0.6}}_{\overline{0.4}})$.

Neither does it hold in general that $E^I_{\mathcal{I}}(A,B)^{\alpha_2}_{\alpha_1}\not\supseteq E(A^{\alpha_2}_{\alpha_1},B^{\overline{1-\alpha_1}}_{\overline{1-\alpha_2}})$. Consider for example the upper border implicator $\mathcal{I}=\mathcal{I}_{\max,\mathcal{N}_s}$, $[\alpha_1,\alpha_2]=[0.3,0.4]$, $A(x)=[0.4,0.5]$ for all $x\in[0,0.5]$ and $A(x)=[0.2,0.3]$ for all $x\in]0.5,1]$, $B(x)=[0.7,0.8]$ for all $x\in[0,0.5]$ and $B(x)=[0.4,0.8]$ for all $x\in]0.5,1]$ and $A(x)=B(x)=0_{L^I}$ for all $x\in\mathbb{R}\backslash[0,1]$.

The binary erosion $E(A^{\alpha_2}_{\alpha_1},B^{\overline{1-\alpha_1}}_{\overline{1-\alpha_2}})$ is then equal to the set $E([0,0.5],[0,0.5])=\{0\}$.

Further, it also holds that:

$$E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)(0)=\inf_{x\in T_0(d_B)}\mathcal{I}_{\max,\mathcal{N}_s}(B(x),A(x))=$$
$$\inf(\inf_{x\in[0,0.5]}\mathcal{I}_{\max,\mathcal{N}_s}(B(x),A(x)),$$
$$\inf_{x\in]0.5,1]}\mathcal{I}_{\max,\mathcal{N}_s}(B(x),A(x)))=\inf([0.4,0.5],[0.2,0.6])=$$
$$[0.2,0.5]\not\geq_{L^I}[\alpha_1,\alpha_2].$$

We conclude that $E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}_{\alpha_1}\not\supseteq E(A^{\alpha_2}_{\alpha_1},B^{\overline{1-\alpha_1}}_{\overline{1-\alpha_2}})$. ♢

For the upper border implicator $\mathcal{I}=\mathcal{I}_{\max,\mathcal{N}_s}$, we have the following partial result.

Proposition 5. *Let $A,B\in\mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}$ that:*

$$E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}_{\alpha_1}\subseteq E(A^{\alpha_2}_{\alpha_1},B^{\overline{1-\alpha_1}}_{\overline{1-\alpha_2}}).$$

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Proof. Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$. It holds that:

$$\begin{aligned}
& y \in E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}) \\
\Leftrightarrow\ & T_y(B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}) \subseteq A_{\alpha_1}^{\alpha_2} \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& ((B_1(x-y) > 1-\alpha_2 \text{ and } B_2(x-y) > 1-\alpha_1) \\
& \Rightarrow (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B))((B_1(x-y) \leq 1-\alpha_2 \text{ or } B_2(x-y) \\
& \leq 1-\alpha_1) \text{ or } (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& ((1-B_1(x-y) \geq \alpha_2 \text{ or } 1-B_2(x-y) \geq \alpha_1) \text{ or} \\
& (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftarrow\ & (\forall x \in T_y(d_B))(\max(1-B_2(x-y), A_1(x)) \geq \alpha_1 \\
& \text{and } \max(1-B_1(x-y), A_2(x)) \geq \alpha_2) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& (\mathcal{I}_{\max,\mathcal{N}_s}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2]) \\
\Leftrightarrow\ & \inf_{x \in T_y(d_B)} \mathcal{I}_{\max,\mathcal{N}_s}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2] \\
\Leftrightarrow\ & E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A, B)(y) \geq_{L^I} [\alpha_1, \alpha_2] \\
\Leftrightarrow\ & y \in E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A, B)_{\alpha_1}^{\alpha_2}
\end{aligned}$$

□

As already illustrated in Example 2, the reverse inclusion does not hold.

Remark that Proposition 5 remains valid in the discrete framework.

3.3.2 Decomposition by weak sub- and supercuts

Proposition 6. *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds that*
(i) $(\forall \alpha_1 \in]0,1])(E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A, B)_{\alpha_1} = E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}))$,
(ii) $(\forall \alpha_2 \in]0,1])(E^I_{\mathcal{I}_{\max,\mathcal{N}_s}}(A, B)^{\alpha_2} = E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}))$.

Proof. Analogous to the proof of Proposition 5. Only now it also holds that

$$(\forall x \in T_y(d_B))(1-B_2(x-y) \geq \alpha_1 \text{ or } A_1(x) \geq \alpha_1)$$
$$\Updownarrow$$
$$(\forall x \in T_y(d_B))(\max(1-B_2(x-y), A_1(x)) \geq \alpha_1)$$

Analogously for the decomposition by weak supercuts. □

Proposition 7. *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\mathcal{I}$ be an upper border implicator on L^I, then it holds that*
(i) $(\forall \alpha_1 \in]0,1])(E^I_{\mathcal{I}}(A, B)_{\alpha_1} \supseteq E(A_{\alpha_1}, B_{\overline{1-\alpha_1}}))$,
(ii) $(\forall \alpha_2 \in]0,1])(E^I_{\mathcal{I}}(A, B)^{\alpha_2} \supseteq E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}))$.

Proof. The proof is completely analogous to the one from proposition 6. We only have that due to lemma 2

$$\inf_{x \in T_y(d_B)} \mathcal{I}_{\max,\mathcal{N}_s}(B_2(x-y), A_1(x))_1 \geq \alpha_1$$
$$\Downarrow$$
$$\inf_{x \in T_y(d_B)} \mathcal{I}(B_2(x-y), A_1(x))_1 \geq \alpha_1$$

only holds in one direction for an arbitrary upper border implicator $\mathcal{I}$ on L^I. Analogously for the decomposition by weak supercuts. □

The reverse inclusion does not hold in general, as illustrated in the first part of Example 2, where replacing the weak $[\alpha_1, \alpha_2]$-cuts by the coinciding weak sub- or supercuts does not affect the results.

Further, the two above properties remain valid in the discrete framework.

3.4 Decomposition of the Interval-valued Fuzzy Closing and Opening

In what follows, we will need the following lemma:

Lemma 3. *Let $A \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $[\alpha_1, \alpha_2] \in L^I$, then it holds that:*
(i) $\alpha_2 \in]0, 0.5] \Rightarrow A^{\alpha_2} \supseteq A^{\overline{\alpha_2}} \supseteq A_{\overline{1-\alpha_2}}$,
(ii) $\alpha_1 \in]0.5, 1] \Rightarrow A_{\alpha_1} \subseteq A^{\overline{1-\alpha_1}}$.

Proof. As an example we prove (i). Suppose that $x \in A_{\overline{1-\alpha_2}}$ and $\alpha_2 \in]0, 0.5]$. The latter implies that $1-\alpha_2 \geq \alpha_2$. Further, since $x \in A_{\overline{1-\alpha_2}}$, we also have that $A_1(x) > 1-\alpha_2$. If we combine the above with the fact that $A_2(x) \geq A_1(x)$, then we find that $A_2(x) > \alpha_2$ and consequently also $A_2(x) \geq \alpha_2$.

(ii) follows in an analogous way. □

3.4.1 Decomposition by weak sub- and supercuts

Proposition 8. *Let $\mathcal{I}$ be an upper border implicator on L^I and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $\alpha_1 \in]0,1]$ that*
(i) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)_{\alpha_1} \supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}})$,
(ii) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)_{\alpha_1} \supseteq D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1})$,
and for all $\alpha_2 \in]0,1]$ that
(iii) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}})$,
(iv) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq D(E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}), -B^{\alpha_2})$.

Proof. As an example we prove (i). Let $\mathcal{I}$ be an upper border implicator on L^I, let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\alpha_1, \alpha_2 \in]0,1]$. From respectively Proposition 6, Proposition 2, and because the binary erosion is increasing in its first argument, we have that:

$$\begin{aligned}
C^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)_{\alpha_1} &= E^I_{\mathcal{I}}(D^I_{\mathcal{C}_{\min}}(A, B), -B)_{\alpha_1} \\
&\supseteq E(D^I_{\mathcal{C}_{\min}}(A, B)_{\alpha_1}, -B^{\overline{1-\alpha_1}}) \\
&\supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}}).
\end{aligned}$$

(ii), (iii) and (iv) follow in an analogous way. □

Under the restriction of $\alpha_2 \in]0, 0.5]$, the above result leads to the following relationships between the weak α_2-supercut of the interval-valued fuzzy closing and opening and the binary counterparts.

Proposition 9. *Let $\mathcal{I}$ be an upper border implicator on L^I and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for $\alpha_2 \in]0, 0.5]$ that*
(i) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq C(A^{\alpha_2}, B^{\alpha_2})$,
(ii) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq C(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$,
and
(iii) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq O(A^{\alpha_2}, B^{\alpha_2})$,
(iv) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A, B)^{\alpha_2} \supseteq O(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$.

Proof. As an example, we prove (i). Let $\mathcal{I}$ be an upper border implicator on L^I, let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\alpha_2 \in]0, 0.5]$.

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From proposition 8, lemma 3 and the fact that the binary erosion is decreasing in its second argument, it follows that:

$$\begin{aligned} C^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2} &\supseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}}) \\ &\supseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B^{\alpha_2}) \\ &= C(A^{\alpha_2},B^{\alpha_2}). \end{aligned}$$

(ii), (iii) and (iv) follow in an analogous way. □

The above results for weak sub- and supercuts remain valid in the discrete framework. Because of the new decomposition properties of the interval-valued fuzzy dilation in the discrete framework compared to the continuous framework (Proposition 3 and 4), also a new property holds for the discrete interval-valued fuzzy closing and opening.

Proposition 10. *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and $\mathcal{I}$ an upper border implicator on $L^I_{r,s}$ and let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_1\in]0,1]\cap I_r$ that*

(i) $C^I_{\mathcal{C}_{\min},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}=E(D(A_{\alpha_1},B_{\alpha_1}),-B^{\overline{1-\alpha_1}})$,
(ii) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)_{\alpha_1}\supseteq E(D(A_{\alpha_1},B_{\alpha_1}),-B^{\overline{1-\alpha_1}})$,
(iii) $C^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq E(D(A_{\alpha_1},B_{\alpha_1}),-B^{\overline{1-\alpha_1}})$,
(iv) $O^I_{\mathcal{C}_{\min},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}=D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$,
(v) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)_{\alpha_1}\supseteq D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$,
(vi) $O^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$,

and for all $\alpha_2\in]0,1]\cap I_s$ that

(vii) $C^I_{\mathcal{C}_{\min},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}=E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}})$,
(viii) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}})$,
(ix) $C^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}\subseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}})$,
(x) $O^I_{\mathcal{C}_{\min},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}=D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$,
(xi) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$,
(xii) $O^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)^{\alpha_2}\subseteq D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$.

Proof. Follows in an analogous way as in the proof of Proposition 8. □

Under the restriction of $\alpha_1\in]0.5,1]\cap I_r$, the above result leads to the following relationships between the weak α_1-subcut of the interval-valued fuzzy closing and opening and the binary counterparts.

Proposition 11. *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_1\in]0.5,1]\cap I_r$ that*

(i) $C^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq C(A_{\alpha_1},B_{\alpha_1})$,
(ii) $C^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq C(A_{\alpha_1},B^{\overline{1-\alpha_1}})$,
(iii) $O^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq O(A_{\alpha_1},B_{\alpha_1})$,
(iv) $O^I_{\mathcal{C},\mathcal{I}_{\max,\mathcal{N}_s}}(A,B)_{\alpha_1}\subseteq O(A_{\alpha_1},B^{\overline{1-\alpha_1}})$,

Proof. Follows in an analogous way as in the proof of Proposition 9. □

Under the restriction of $\alpha_2\in]0,0.5]\cap I_s$, we come to the same results as in the continuous case (Proposition 9).

Proposition 12. *Let $\mathcal{I}$ be an upper border implicator on $L^I_{r,s}$ and let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_2\in]0,0.5]\cap I_s$ that*

(i) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq C(A^{\alpha_2},B^{\alpha_2})$,
(ii) $C^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq C(A^{\alpha_2},B_{\overline{1-\alpha_2}})$,
(iii) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq O(A^{\alpha_2},B^{\alpha_2})$,
(iv) $O^I_{\mathcal{C}_{\min},\mathcal{I}}(A,B)^{\alpha_2}\supseteq O(A^{\alpha_2},B_{\overline{1-\alpha_2}})$.

Proof. Follows in an analogous way as in the proof of Proposition 9. □

4 Conclusion

In this paper we have introduced the basic interval-valued fuzzy morphological operations and we have studied their decomposition by weak $[\alpha_1,\alpha_2]$-cuts both in the general continuous case and the practical discrete case. We found out that a few properties that do not hold in the continuous case do hold in the practical discrete case. Finally, we would like to mention that similar, but not completely analogous results hold for the decomposition by strict $[\alpha_1,\alpha_2]$-cuts. Those results will be published elsewhere.

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References

[1] J. Serra. *Image Analysis and Mathematical Morphology*, Academic Press, London, 1982.

[2] R.M. Haralick, R.S. Sternberg and X. Zhuang. Image analysis using mathematical morphology, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 9(4):532-550, 1987.

[3] B. De Baets. Fuzzy morphology: a logical approach, *Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics, and Neural Network Approach* (Ayyub, B.M. & Gupta, M.M., editors), Kluwer Academic Publishers, Boston, pp. 53-67, 1997.

[4] P. Sussner and M.E. Valle: Classification of fuzzy mathematical morphologies based on concepts of inclusion measure and duality, *Journal of Mathematical Imaging and Vision* , 32(2):139-159, October 2008.

[5] I. Bloch. Mathematical Morphology on Bipolar Fuzzy Sets, *Proceedings of ISMM 2007 (International Symposium on Mathematical Morphology)*, pp. 3-4, 2007.

[6] M. Nachtegael, P. Sussner, T. Mélange and E.E. Kerre. Modelling numerical and spatial uncertainty in grayscale image capture using fuzzy set theory, *Proceedings of NASTEC 2008*, 2008.

[7] L. Zadeh. Fuzzy Sets, *Information and Control*, 8:338-353, 1965.

[8] R. Sambuc. *Fonctions Φ-floues. Application à l'aide au diagnostic en pathologie thyroidienne*, Ph.D. thesis, Univ. Marseille, France, 1975.

[9] G. Deschrijver and E.E. Kerre. On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems*, 133:227-235, 2003.

[10] G. Deschrijver and E.E. Kerre. On the cuts of intuitionistic fuzzy compositions, *Kuwait Journal of Science & Engineering*, 32(1):17-38, 2005.

[11] M. Nachtegael and E.E. Kerre. Connections between binary, gray-scale and fuzzy mathematical morphology, *Fuzzy Sets and Systems*, 124:73-86, 2001.

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