Non-conjunctive and non-disjunctive uninorms in Atanassov's intuitionistic fuzzy set theory

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Abstract— *Uninorms are a generalization of t-norms and t-conorms for which the neutral element is an element of $[0,1]$ which is not necessarily equal to 0 (as for t-norms) or 1 (as for t-conorms). Uninorms on the unit interval are either conjunctive or disjunctive, i.e. they aggregate the pair $(0,1)$ to either 0 or 1. In real-life applications, this kind of aggregation may be counter-intuitive. Atanassov's intuitionistic fuzzy set theory is an extension of fuzzy set theory which allows to model uncertainty about the membership degrees. In Atanassov's intuitionistic fuzzy set theory there exist uninorms which are neither conjunctive nor disjunctive. In this paper we study such uninorms more deeply and we investigate the structure of these uninorms. We also give several examples of uninorms which are neither conjunctive nor disjunctive.*

Keywords— Conjunctive, disjunctive, interval-valued fuzzy set, intuitionistic fuzzy set, uninorm.

1 Introduction

Interval-valued fuzzy set theory [1, 2] is an extension of fuzzy set theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree (using interval-valued fuzzy sets is not always the best approach to deal with uncertainty, see [3] for more information). Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [4]. In [5] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to L-fuzzy set theory in the sense of Goguen [6] w.r.t. a special lattice $\mathcal{L}^I$.

Uninorms are an important generalization of t-norms and t-conorms introduced by Yager and Rybalov [7]. Uninorms allow for a neutral element lying anywhere in the unit interval rather than at one or zero as is the case for t-norms and t-conorms. Uninorms on the unit interval are either conjunctive or disjunctive, i.e. they aggregate the pair $(0,1)$ to either 0 or 1. In real-life applications, this kind of aggregation may be counter-intuitive, e.g. in customer satisfaction modelling, if an aspect of the product receives a negative evaluation and another aspect a positive evaluation, then in general the global evaluation will neither be very negative or very positive, but rather be quite uncertain. This situation can be modelled by using uninorms in Atanassov's intuitionistic fuzzy set theory, which can be neither conjunctive nor disjunctive (see [8]). In this paper we therefore investigate such uninorms more deeply.

Definition 1.1 *We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where*

$$L^I = \{[x_1,x_2] \mid (x_1,x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\}, \qquad (1)$$

$$[x_1,x_2] \leq_{L^I} [y_1,y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \qquad (2)$$

for all $[x_1,x_2], [y_1,y_2]$ in L^I.

Similarly as Lemma 2.1 in [5] it can be shown that $\mathcal{L}^I$ is a complete lattice.

Definition 1.2 [1, 2] *An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.*

Definition 1.3 [4] *An intuitionistic fuzzy set on U is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}, \qquad (3)$$

where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$\begin{array}{rcl} A : U & \to & L^I : \\ u & \mapsto & [\mu_A(u), 1-\nu_A(u)], \end{array} \qquad (4)$$

In Figure 1 the set L^I is shown. Note that each $x = [x_1,x_2] \in L^I$ is represented by the point $(x_1,x_2) \in \mathbb{R}^2$.

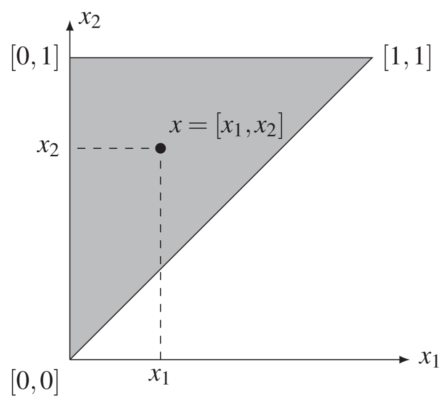

Figure 1: The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e. $x = [x_1,x_2]$. The smallest and the largest element of $\mathcal{L}^I$

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are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. Note that, for x, y in L^I, $x <_{L^I} y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. If for x, y in L^I it holds that either $x_1 < y_1$ and $x_2 > y_2$, or $x_1 > y_1$ and $x_2 < y_2$, then x and y are incomparable w.r.t. $\leq_{L^I}$, denoted as $x \|_{L^I} y$. We define for further usage the set $D = \{[x_1, x_1] \mid x_1 \in [0,1]\}$.

Definition 1.4 *A t-norm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{T} : (L^I)^2 \to L^I$ which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.*

A t-conorm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{S} : (L^I)^2 \to L^I$ which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.

Definition 1.5 *A negation on $\mathcal{L}^I$ is a decreasing mapping $\mathcal{N} : L^I \to L^I$ for which $\mathcal{N}(0_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$ and $\mathcal{N}(1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^I$, then $\mathcal{N}$ is called involutive.*

Let N be a negation on $([0,1], \leq)$. Then the mapping $\mathcal{N}_N : L^I \to L^I$ defined by, for all $x \in L^I$,

$$\mathcal{N}_N(x) = [N(x_2), N(x_1)], \tag{5}$$

is a negation on $\mathcal{L}^I$.

We will also need the following result and definition (see [9, 10, 11, 12, 13]).

Theorem 1.1 *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms on $([0,1], \leq)$ and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined by, for all x, y in $[0,1]$,*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right), \\ \quad \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y), \quad \text{otherwise}, \end{cases} \tag{6}$$

is a t-norm on $([0,1], \leq)$.

Definition 1.6 *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms on $([0,1], \leq)$ and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T defined by (6) is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we will write*

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}. \tag{7}$$

2 Uninorms on $\mathcal{L}^I$

The following definition of a uninorm on $\mathcal{L}^I$ is a straightforward generalization of the definition of a uninorm on the unit interval introduced by Yager and Rybalov [7, 14].

Definition 2.1 [8] *A uninorm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{U} : (L^I)^2 \to L^I$ for which there exists an $e \in L^I$ such that $\mathcal{U}(e, x) = x$, for all $x \in L^I$. The element e is called the neutral element of $\mathcal{U}$.*

For any uninorm U on the unit interval, there exist increasing bijections $\phi_e : [0,e] \to [0,1]$ and $\psi_e : [e,1] \to [0,1]$ with increasing inverse, a t-norm T_U and a t-conorm S_U on $([0,1], \leq)$ such that [14]

(i) $(\forall (x,y) \in [0,e]^2)(U(x,y) = \phi_e^{-1}(T_U(\phi_e(x), \phi_e(y))))$;

(ii) $(\forall (x,y) \in [e,1]^2)(U(x,y) = \psi_e^{-1}(S_U(\psi_e(x), \psi_e(y))))$.

Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in L^I$. We define $E = \{x \mid x \in L^I \text{ and } x \leq_{L^I} e\}$ and $E' = \{x \mid x \in L^I \text{ and } x \geq_{L^I} e\}$. In [8] it is shown that if $e \notin D$, then there does not exist increasing bijections $\Phi_e : E \to L^I$ and $\Psi_e : E' \to L^I$ such that Φ_e^{-1} and Ψ_e^{-1} are increasing. On the other hand, if $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, then the mappings $\Phi_e : E \to L^I$ and $\Psi_e : E' \to L^I$ defined by, for all $x \in L^I$,

$$\Phi_e(x) = \left[\frac{x_1}{e_1}, \frac{x_2}{e_1}\right], \tag{8}$$

$$\Psi_e(x) = \left[\frac{x_1 - e_1}{1 - e_1}, \frac{x_2 - e_1}{1 - e_1}\right]. \tag{9}$$

are increasing bijections for which the inverse is also increasing. As a consequence, the above result can only be extended if $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$.

From now on, we denote for any t-norm T and t-conorm S on $([0,1], \leq)$, $T_{\phi_e} = \phi_e^{-1} \circ T \circ (\phi_e \times \phi_e)$ and $S_{\psi_e} = \psi_e^{-1} \circ S \circ (\psi_e \times \psi_e)$, where $\times$ denotes the product operation [15]. A similar notation will be used for t-(co)norms and bijections on $\mathcal{L}^I$.

Theorem 2.1 [8] *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Then:*

(i) the mapping $\mathcal{T}_\mathcal{U} : (L^I)^2 \to L^I$ defined by, for all $x, y \in L^I$,

$$\mathcal{T}_\mathcal{U}(x,y) = \Phi_e(\mathcal{U}(\Phi_e^{-1}(x), \Phi_e^{-1}(y))) \tag{10}$$

is a t-norm on $\mathcal{L}^I$;

(ii) the mapping $\mathcal{S}_\mathcal{U} : (L^I)^2 \to L^I$ defined by, for all $x, y \in L^I$,

$$\mathcal{S}_\mathcal{U}(x,y) = \Psi_e(\mathcal{U}(\Psi_e^{-1}(x), \Psi_e^{-1}(y))) \tag{11}$$

is a t-conorm on $\mathcal{L}^I$.

Theorem 2.2 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Then for all x, y in L^I,*

$$x \leq_{L^I} e \leq_{L^I} y \implies \inf(x,y) \leq_{L^I} \mathcal{U}(x,y) \leq_{L^I} \sup(x,y). \tag{12}$$

These properties show that uninorms are well suited to model human evaluations (e.g. customer satisfaction). Customers which evaluate the performance of all aspects of a certain product high, have a tendency to give the global satisfaction degree an even higher value; on the other hand customers which globally consider the performance of the various aspects as insufficient, will give a low global evaluation. So we observe “reinforcement”: a collection of high (low) rates “reinforce” each other and yield a global evaluation rate that is even higher (resp. lower) than each individual rate. If, however, a customer gives high scores only to some aspects and low scores for other aspects, then the global score will in general be located between the lowest and the highest value. This is “compensation”. From Theorem 2.1 it follows that $\mathcal{U}\big|_{E^2}$ behaves like a t-norm, in particular $\mathcal{U}(x,y) \leq_{L^I} \inf(x,y)$, for all x, y in E. On the other hand, $\mathcal{U}\big|_{E'^2}$ behaves like a t-conorm, so $\mathcal{U}(x,y) \geq_{L^I} \sup(x,y)$, for all x, y in E'. Finally, if $x \leq_{L^I} e$ and $y \geq_{L^I} e$ (or conversely), then $\mathcal{U}(x,y)$ is a number between

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$\inf(x,y)$ and $\sup(x,y)$. So, clearly, uninorms show a reinforcing behaviour on E^2 and E'^2, and a compensating behaviour on $E \times E'$ and $E' \times E$ (see [16, 17, 18, 19] for more details).

For uninorms on the unit interval, however, $U(0,1)$ can only have two values: 0 or 1 (see [14]). In the first case the uninorm is called "conjunctive" and in the second case "disjunctive". However, in both cases the compensatory behaviour of the uninorm is violated. For uninorms on $\mathcal{L}^I$ we have the following property.

Theorem 2.3 [8] *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Then either $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$ or $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$ or $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \parallel_{L^I} e$.*

Hence uninorms on $\mathcal{L}^I$ are not necessarily conjunctive or disjunctive. It is possible that a uninorm on $\mathcal{L}^I$ shows compensatory behaviour between $0_{\mathcal{L}^I}$ and $1_{\mathcal{L}^I}$. If one aspect of a product has a very negative evaluation ($0_{\mathcal{L}^I}$) and another aspect is very positively evaluated ($1_{\mathcal{L}^I}$), then in general it will be very difficult to give a global evaluation of the product, in fact the global evaluation will contain a lot of uncertainty. Therefore it makes more sense to use a uninorm $\mathcal{U}$ for which $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \parallel_{L^I} e$.

3 Uninorms on $\mathcal{L}^I$ which are neither conjunctive nor disjunctive

In this section we try to obtain more information about the structure of uninorms which are neither conjunctive nor disjunctive by investigating the possible values of $\mathcal{U}(x,y)$ with x, y in L^I. First we give an example of a uninorm on $\mathcal{L}^I$ that is neither conjunctive nor disjunctive, in order to show that such uninorms do exists.

Example 3.1 *Let for all $e_1 \in]0,1[$, U_{e_1} be the uninorm on $([0,1],\leq)$ defined by, for all x_1, y_1 in $[0,1]$,*

$$U_{e_1}(x_1,y_1) = \begin{cases} \max(x_1,y_1), & \text{if } x_1 \geq e_1 \text{ and } y_1 \geq e_1; \\ \min(x_1,y_1), & \text{else.} \end{cases} \tag{13}$$

Let now, for all x, y in L^I,

$$\mathcal{U}(x,y) = [U_{e_1}(x_1,y_1), 1 - U_{1-e_1}(1-x_2, 1-y_2)]. \tag{14}$$

Then $\mathcal{U}$ is a uninorm on $\mathcal{L}^I$ with neutral element $e = [e_1, e_1]$. Since $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = [0,1]$, $\mathcal{U}$ is neither conjunctive nor disjunctive.

In general, if U_1 is an arbitrary conjunctive uninorm and U_2 an arbitrary disjunctive uninorm on $([0,1],\leq)$, then the mapping $\mathcal{U} : (L^I)^2 \to L^I : (x,y) \mapsto [U_1(x_1,y_1), U_2(x_2,y_2)]$, for all x, y in L^I, is a uninorm on $\mathcal{L}^I$ for which $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = [0,1]$.

Lemma 3.1 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Then, for all $x \in L^I$,*

(i) either $\mathcal{U}(0_{\mathcal{L}^I}, x) = 0_{\mathcal{L}^I}$ or $\mathcal{U}(0_{\mathcal{L}^I}, x) \notin E$,

(ii) either $\mathcal{U}(1_{\mathcal{L}^I}, x) = 1_{\mathcal{L}^I}$ or $\mathcal{U}(1_{\mathcal{L}^I}, x) \notin E'$.

Theorem 3.2 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. If $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \parallel_{L^I} e$, then, for all $x \in L^I$,*

(i) $\mathcal{U}(0_{\mathcal{L}^I}, x) \parallel_{L^I} e$ or $\mathcal{U}(0_{\mathcal{L}^I}, x) = 0_{\mathcal{L}^I}$,

(ii) $\mathcal{U}(1_{\mathcal{L}^I}, x) \parallel_{L^I} e$ or $\mathcal{U}(1_{\mathcal{L}^I}, x) = 1_{\mathcal{L}^I}$.

If one aspect of a product has a negative evaluation $x \in L^I$ with $x \leq_{L^I} e$ and another aspect has a positive evaluation $y \in L^I$ with $y \geq_{L^I} e$, then the global evaluation will be rather neutral and contain some uncertainty. Therefore it is natural to expect that $\mathcal{U}(x,y) \parallel_{L^I} e$. We investigate for which x and y in L^I this is the case.

Lemma 3.3 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Assume that $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \parallel_{L^I} e$.*

(i) Let arbitrarily $x \in E$. If $\mathcal{U}(1_{\mathcal{L}^I}, x) = 1_{\mathcal{L}^I}$, then $\mathcal{U}(1_{\mathcal{L}^I}, [x_1, y_2]) = 1_{\mathcal{L}^I}$, for all $y_2 \in [x_1, e_1]$.

(ii) Let arbitrarily $x \in E'$. If $\mathcal{U}(0_{\mathcal{L}^I}, x) = 0_{\mathcal{L}^I}$, then $\mathcal{U}(0_{\mathcal{L}^I}, [y_1, x_2]) = 0_{\mathcal{L}^I}$, for all $y_1 \in [e_1, x_2]$.

Theorem 3.4 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. If $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \parallel_{L^I} e$, then*

(i) there exists an $\alpha \in D \cap E$ such that (see Figure 2)

- *$\mathcal{U}(1_{\mathcal{L}^I}, x) \parallel_{L^I} e$ for all $x \in L^I$ satisfying $x_1 < \alpha_1$ and $x_2 \leq e_1$, and*
- *$\mathcal{U}(1_{\mathcal{L}^I}, x) = 1_{\mathcal{L}^I}$, for all $x \in L^I$ satisfying $x_1 > \alpha_1$,*

(ii) there exists a $\beta \in D \cap E'$ such that

- *$\mathcal{U}(0_{\mathcal{L}^I}, x) \parallel_{L^I} e$ for all $x \in L^I$ satisfying $x_1 \geq e_1$ and $x_2 > \beta_1$, and*
- *$\mathcal{U}(0_{\mathcal{L}^I}, x) = 0_{\mathcal{L}^I}$, for all $x \in L^I$ satisfying $x_2 < \beta_1$.*

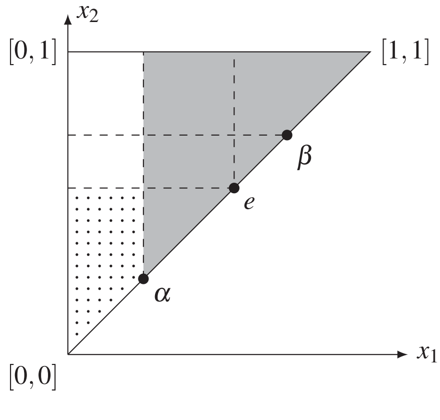

Figure 2: The grey area is the set of elements x for which $\mathcal{U}(x, 1_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$, the dotted area is the set of elements x for which $\mathcal{U}(x, 1_{\mathcal{L}^I}) \parallel_{L^I} e$.

Example 3.2 We give an example of a uninorm which satisfies the results in Theorem 3.4 for a non-trivial α and β. Let arbitrarily $\alpha \in D \cap E \setminus \{0_{\mathcal{L}^I}, e\}$ and $\beta \in D' \cap E' \setminus \{e, 1_{\mathcal{L}^I}\}$. Let T_{1a} and T_{1b} be arbitrary t-norms, S_{1a} and S_{1b} arbitrary t-conorms on $([0,1],\leq)$, and define

$$T_1 = (\langle 0, \phi_e(\alpha_1), T_{1a} \rangle, \langle \phi_e(\alpha_1), 1, T_{1b} \rangle), \tag{15}$$
$$S_2 = (\langle 0, \psi_e(\beta_1), S_{2a} \rangle, \langle \psi_e(\beta_1), 1, S_{2b} \rangle). \tag{16}$$

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Define the mappings $U_1 : [0,1]^2 \to [0,1]$ and $U_2 : [0,1]^2 \to [0,1]$ by, for all x_1, y_1, x_2, y_2 in $[0,1]$,

$$U_1(x_1,y_1) = \begin{cases} (T_1)_{\phi_e}(x_1,y_1), \text{ if } \max(x_1,y_1) \le e_1, \\ (S_1)_{\psi_e}(x_1,y_1), \text{ if } \min(x_1,y_1) \ge e_1, \\ 1, \text{ if } (x_1 > \alpha_1 \text{ and } y_1 = 1) \\ \quad \text{ or } (y_1 > \alpha_1 \text{ and } x_1 = 1), \\ \min(x_1,y_1), \text{ else,} \end{cases} \tag{17}$$

$$U_2(x_2,y_2) = \begin{cases} (T_2)_{\phi_e}(x_2,y_2), \text{ if } \max(x_2,y_2) \le e_1, \\ (S_2)_{\psi_e}(x_2,y_2), \text{ if } \min(x_2,y_2) \ge e_1, \\ 0, \text{ if } (x_2 < \beta_1 \text{ and } y_2 = 0) \\ \quad \text{ or } (y_2 < \beta_1 \text{ and } x_2 = 0), \\ \max(x_2,y_2), \text{ else,} \end{cases} \tag{18}$$

Then U_1 is a conjunctive uninorm and U_2 is a disjunctive uninorm on $([0,1],\le)$. The mapping $\mathcal{U} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,

$$\mathcal{U}(x,y) = [U_1(x_1,y_1), U_2(x_2,y_2)], \tag{19}$$

is a uninorm on $\mathcal{L}^I$ for which $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = [0,1]$ and for which the results in Theorem 3.4 hold for the given α and β.

From now on α and β will be the elements of $\mathcal{L}^I$ introduced in Theorem 3.4.

Lemma 3.5 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. If $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \,\|_{L^I}\, e$, then for all $x \in E$ and $y \in E'$ satisfying $x_1 < \alpha_1$ and $y_2 > \beta_1$ it holds that $\mathcal{U}(x,y) \,\|_{L^I}\, e$.*

Theorem 3.6 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. If $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \,\|_{L^I}\, e$, then for all $x \in E$ and $y \in E'$ satisfying $x_1 < \alpha_1$ and $y_2 > \beta_1$ it holds that $(\mathcal{U}(x,y))_1 \le \alpha_1$ and $(\mathcal{U}(x,y))_2 \ge \beta_1$.*

Corollary 3.7 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Assume that $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \,\|_{L^I}\, e$.*

(i) Let arbitrarily $a = [\alpha_1, a_2] \in E$ and $y \in E'$ such that $y_2 > \beta_1$. Then
$$\lim_{\substack{x \to a \\ x_1 < \alpha_1}} (\mathcal{U}(x,y))_1 = \alpha_1. \tag{20}$$

(ii) Let arbitrarily $b = [b_1, \beta_2] \in E'$ and $x \in E$ such that $x_1 < \alpha_1$. Then
$$\lim_{\substack{y \to b \\ y_2 > \beta_2}} (\mathcal{U}(x,y))_2 = \beta_2. \tag{21}$$

In the above, the limits are calculated using on L^I the Euclidean metric function $d^E(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, for all x, y in L^I.

Theorem 3.8 *Let $\mathcal{U}$ be a uninorm on $\mathcal{L}^I$ with neutral element $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$. Assume that $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) \,\|_{L^I}\, e$.*

(i) For all $x \in E$ and $y \in E'$ satisfying $x_1 > \alpha_1$ and $y_2 > \beta_1$ it holds that $\mathcal{U}(x,y) \ge_{L^I} [\alpha_1, \beta_1]$.

(ii) For all $x \in E$ and $y \in E'$ satisfying $x_1 < \alpha_1$ and $y_2 < \beta_1$ it holds that $\mathcal{U}(x,y) \le_{L^I} [\alpha_1, \beta_1]$.

4 The value of $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I})$

In this section we check which are the possible values for $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I})$ in the case that $\mathcal{U}$ is neither conjunctive nor disjunctive.

Lemma 4.1 *For any $\alpha \in L^I$ and $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ such that $\alpha \,\|_{L^I}\, e$, $\alpha_1 > 0$ and $\alpha_2 < 1$, there exists an involutive negation N on $([0,1],\le)$ such that $N(\alpha_1) = \alpha_2$ and $N(e_1) = e_1$.*

Theorem 4.2 *Let $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, $\alpha \in L^I$, T_1 and T_2 be t-norms, S_1 and S_2 be t-conorms on $([0,1],\le)$ such that*

(i) $\alpha \,\|_{L^I}\, e$,

(ii) there exist t-norms T_{1a} and T_{1b} on $([0,1],\le)$ such that $T_1 = (\langle 0, \phi_e(\alpha_1), T_{1a}\rangle, \langle \phi_e(\alpha_1), 1, T_{1b}\rangle)$,

(iii) there exist t-conorms S_{2a} and S_{2b} on $([0,1],\le)$ such that $S_2 = (\langle 0, \psi_e(\alpha_2), S_{2a}\rangle, \langle \psi_e(\alpha_2), 1, S_{2b}\rangle)$,

(iv) $T_1(x_1,y_1) \le T_2(x_1,y_1)$ and $S_1(x_1,y_1) \le S_2(x_1,y_1)$, for all x_1, y_1 in $[0,1]$.

Define the mapping $\mathcal{U} : (L^I)^2 \to L^I$ by, for all x, y in L^I,

$$(\mathcal{U}(x,y))_1 = \begin{cases} \alpha_1, \textit{ if } (x_1 < \alpha_1 \textit{ and } y_1 \ge \alpha_1 \textit{ and } y_2 > e_1) \\ \quad \textit{ or } (y_1 < \alpha_1 \textit{ and } x_1 \ge \alpha_1 \textit{ and } x_2 > e_1), \\ U_1(x_1,y_1), \textit{ else,} \end{cases} \tag{22}$$

$$(\mathcal{U}(x,y))_2 = \begin{cases} \alpha_2, \textit{ if } (x_2 > \alpha_2 \textit{ and } y_2 \le \alpha_2 \textit{ and } y_1 < e_1) \\ \quad \textit{ or } (y_2 > \alpha_2 \textit{ and } x_2 \le \alpha_2 \textit{ and } x_1 < e_1), \\ U_2(x_2,y_2), \textit{ else.} \end{cases} \tag{23}$$

where, for all x_1, y_1, x_2, y_2 in $[0,1]$,

$$U_1(x_1,y_1) = \begin{cases} (T_1)_{\phi_e}(x_1,y_1), & \textit{if } \max(x_1,y_1) \le e_1, \\ (S_1)_{\psi_e}(x_1,y_1), & \textit{if } \min(x_1,y_1) \ge e_1, \\ \min(x_1,y_1), & \textit{else,} \end{cases} \tag{24}$$

$$U_2(x_2,y_2) = \begin{cases} (T_2)_{\phi_e}(x_2,y_2), & \textit{if } \max(x_2,y_2) \le e_1, \\ (S_2)_{\psi_e}(x_2,y_2), & \textit{if } \min(x_2,y_2) \ge e_1, \\ \max(x_2,y_2), & \textit{else.} \end{cases} \tag{25}$$

Then $\mathcal{U}$ is a uninorm on $\mathcal{L}^I$ with neutral element e for which $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = \alpha$.

Theorem 4.2 shows that for any $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ and any $\alpha \in L^I$ such that $\alpha \,\|_{L^I}\, e$, there exists a uninorm $\mathcal{U}$ on $\mathcal{L}^I$ with neutral element e such that $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = \alpha$.

In the following theorem we show that for most values of $\alpha \in L^I$ such that $\alpha \,\|_{L^I}\, e$, it is even possible to find uninorms satisfying $\mathcal{U}(0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) = \alpha$, which are self-dual.

Theorem 4.3 *Let $e \in D \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$, $\alpha \in L^I$, T be a t-norm, S a t-conorm and N a negation on $([0,1],\le)$ such that*

(i) $\alpha \,\|_{L^I}\, e$ and either $\alpha \,\|_{L^I}\, [0,1]$ or $\alpha = [0,1]$,

(ii) N is involutive, $N(\alpha_1) = \alpha_2$ and $N(e_1) = e_1$,

(iii) there exist t-norms T_a and T_b on $([0,1],\le)$ such that $T = (\langle 0, \phi_e(\alpha_1), T_a\rangle, \langle \phi_e(\alpha_1), 1, T_b\rangle)$,

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(iv) $T_{\phi_e}(x_1,y_1) \leq N(S_{\psi_e}(N(x_1),N(y_1)))$, *for all* x_1 *and* y_1 *in* $[0,e_1]$.

Define the mapping $\mathcal{U} : (L^I)^2 \to L^I$ *by, for all* x,y *in* L^I,

$$(\mathcal{U}(x,y))_1 = \begin{cases} \alpha_1, \text{ if } (x_1 < \alpha_1 \text{ and } y_1 \geq \alpha_1 \text{ and } y_2 > e_1) \\ \quad \text{or } (y_1 < \alpha_1 \text{ and } x_1 \geq \alpha_1 \text{ and } x_2 > e_1), \\ U(x_1,y_1), \text{ else}, \end{cases} \tag{26}$$

$$(\mathcal{U}(x,y))_2 = N((\mathcal{U}(\mathcal{N}_N(x),\mathcal{N}_N(y)))_1). \tag{27}$$

where, for all x_1,y_1 *in* $[0,1]$,

$$U(x_1,y_1) = \begin{cases} T_{\phi_e}(x_1,y_1), & \text{if } \max(x_1,y_1) \leq e_1, \\ S_{\psi_e}(x_1,y_1), & \text{if } \min(x_1,y_1) \geq e_1, \\ \min(x_1,y_1), & \text{else}. \end{cases} \tag{28}$$

Then $\mathcal{U}$ *is a uninorm on* $\mathcal{L}^I$ *with neutral element e for which* $\mathcal{U}(0_{\mathcal{L}^I},1_{\mathcal{L}^I}) = \alpha$ *and, for all* x,y *in* L^I,

$$\mathcal{U}(x,y) = \mathcal{N}_N(\mathcal{U}(\mathcal{N}_N(x),\mathcal{N}_N(y))). \tag{29}$$

Example 4.1 *Let arbitrarily* $e \in D$ *and* $\alpha \in L^I$ *with* $\alpha \,\|_{L^I}\, e$ *and* $\alpha \,\|_{L^I}\, [0,1]$. *Define for all* $x_1 \in [0,1]$,

$$N(x_1) = \begin{cases} 1 - \frac{1-\alpha_2}{\alpha_1}x_1, & \text{if } x_1 \in [0,\alpha_1], \\ e_1 + \frac{\alpha_2-e_1}{\alpha_1-e_1}(x_1-e_1), & \text{if } x_1 \in [\alpha_1,e_1], \\ e_1 + \frac{\alpha_1-e_1}{\alpha_2-e_1}(x_1-e_1), & \text{if } x_1 \in [e_1,\alpha_2], \\ -\frac{\alpha_1}{1-\alpha_2}(x_1-e_1), & \text{if } x_1 \in [\alpha_2,1]. \end{cases} \tag{30}$$

Then N *is an involutive negation with* $N(\alpha_1) = \alpha_2$ *and* $N(e_1) = e_1$. *Define* $T = (\langle 0,\phi_e(\alpha_1),P\rangle,\langle\phi_e(\alpha_1),1,\min\rangle)$, *where P is the product t-norm on the unit interval. Then for all* $(x_1,y_1) \in [0,e_1]^2$,

$$T_{\phi_e}(x_1,y_1) = \begin{cases} \frac{1}{\alpha_1}x_1y_1, & \text{if } (x_1,y_1) \in [0,\alpha_1]^2, \\ \min(x_1,y_1), & \text{else}. \end{cases} \tag{31}$$

Let now for all $(x_1,y_1) \in [e_1,1]^2$, $S_{\psi_e} = N \circ T_{\phi_e} \circ (N \times N)$. *Define U,* $(\mathcal{U}(x,y))_1$ *and* $(\mathcal{U}(x,y))_2$ *in a similar way as in Theorem 4.3. Then* $\mathcal{U}$ *is a uninorm on* $\mathcal{L}^I$ *with neutral element e for which* $\mathcal{U}(0_{\mathcal{L}^I},1_{\mathcal{L}^I}) = \alpha$ *and which is self-dual w.r.t.* $\mathcal{N}_N$.

The question remains whether for any $e \in D \setminus \{0_{\mathcal{L}^I},1_{\mathcal{L}^I}\}$, any $\alpha \in L^I$ such that $\alpha \,\|_{L^I}\, e$, and also any t-norm $\mathcal{T}$ and any t-conorm $\mathcal{S}$ on $\mathcal{L}^I$, there exists a uninorm $\mathcal{U}$ on $\mathcal{L}^I$ with neutral element e such that $\mathcal{U}(0_{\mathcal{L}^I},1_{\mathcal{L}^I}) = \alpha$, $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}$.

5 Conclusion

In this paper we have studied uninorms on the lattice $\mathcal{L}^I$, which is the underlying lattice of both Atanassov's intuitionistic fuzzy set theory and interval-valued fuzzy set theory. Such uninorms $\mathcal{U}$ can be neither conjunctive nor disjunctive, in which case $\mathcal{U}(0_{\mathcal{L}^I},1_{\mathcal{L}^I})$ is an element of L^I which is incomparable to the neutral element of $\mathcal{U}$. We have investigated the value $\mathcal{U}(x,y)$ in the case that x and y are located in certain areas of L^I and we have found several restrictions. For any value of $\alpha \in L^I$ which is incomparable to an arbitrary element e, we have constructed a uninorm $\mathcal{U}$ with neutral element e and for which $\mathcal{U}(0_{\mathcal{L}^I},1_{\mathcal{L}^I}) = \alpha$.

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