Chaos induced by turbulent and erratic functions

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Abstract— *Let (X, d) be a compact metric space and $f : X \to X$ a continuous function and consider the hyperspace $(\mathcal{K}(X), H)$ of all nonempty compact subsets of X endowed with the Hausdorff metric induced by d. Let $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ be defined by $\bar{f}(A) = \{f(a)/a \in A\}$ the natural extension of f to $\mathcal{K}(X)$, then the aim of this work is to study the dynamics of $\bar{f}$ when f is turbulent (erratic, respectively) and its relationships.*

Keywords— Chaos, dynamical systems, turbulent functions, erratic functions.

1 Introduction

Let (X, d) be a compact metric space and $f : X \to X$ a continuous function. If we consider the extension $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$, then an interesting problem is to analyze the chaotic relations between f and $\bar{f}$.

In this direction, Román-Flores in [1] proved that transitivity of $\bar{f}$ implies transitivity of f, whereas in [2] the author proved that sensitivity of $\bar{f}$ implies sensitivity of f and periodic density of f implies periodic density of $\bar{f}$. Also, Banks in [3] shows that transitivity of $\bar{f}$ is equivalent to f weakly mixing. Several other chaotic relations between f and $\bar{f}$ have also been investigated by many authors, for example Cánovas et al. [4] investigated some connections in relation to topological entropy, Gu in [5] studied Kato's chaos, and Ma et al. [6] investigated some aspects of topological entropy, Li-Yorke chaos and distributional chaos.

The aim of this work is to explore some turbulent and erratic connections between f and $\bar{f}$.

2 Preliminaries

Let (X, d) be a compact metric space and let $\mathcal{K}(X)$ be the class of all non-empty and compact subsets of X. If $A \in \mathcal{K}(X)$ we define the "ϵ-neighbourhood of A" as the set

$$N(A, \epsilon) = \{x \in X/\ d(x, A) < \epsilon\},$$

where $d(x, A) = \inf_{a \in A} d(x, a)$.

The Hausdorff metric on $\mathcal{K}(X)$ is defined as

$$H(A, B) = \inf\{\epsilon > 0/\ A \subseteq N(B, \epsilon)\ and\ B \subseteq N(A, \epsilon)\},$$

and it is well known that $(\mathcal{K}(X), H)$ is a compact metric space (see [7]).

If $A \in \mathcal{K}(X)$ we denote by $B(A, \epsilon)$ the ball centered in A and radius ϵ in H-metric.

Remark 1 *An equivalent formula for H is given by*

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\right\}.$$

Remark 2 *If K_n, K_0 are nonemty compact subsets of X such that $K_n \xrightarrow{H} K_0$, then (see [7])*

$$K_0 = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n \geq p} K_n}. \tag{1}$$

If X has a real linear structure then, by using the Minkowski sum between two sets, a linear structure of convex cone is defined on $\mathcal{K}(X)$ by means

$$A + B = \{a + b,\ a \in A, b \in B\}\ \text{and}\ \lambda A = \{\lambda a,\ a \in A\},$$

for all $A, B \in \mathcal{K}(X),\ \lambda \in \mathbb{R}$.

Let A be a subset of X. Then we define the extension of A to $\mathcal{K}(X)$ as

$$e(A) = \{K \in \mathcal{K}(X)/\ K \subseteq A\}.$$

The following results were proved by the author in [1].

Lemma 1 *Let A, B be two subsets of X and let $f : X \to X$ be a continuous function. Then,*

i) $e(A) = \emptyset$ *if and only if* $A = \emptyset$.

ii) $e(A \cap B) = e(A) \cap e(B)$

iii) *If A is a nonempty open subset of X, then $e(A)$ is a nonempty open subset of $\mathcal{K}(X)$.*

iv) $\bar{f}(e(A)) \subseteq e(f(A))$

v) $\bar{f}^p = \overline{f^p}$ *, for every $p \in \mathbb{N}$.*

We remark that if f is a bijective map then

$$\bar{f}(e(A)) = e(f(A)). \tag{2}$$

Also, as a direct consequence of (1) we have

Lemma 2 *If A is a nonempty closed subset of X, then $e(A)$ is a nonempty closed subset of $\mathcal{K}(X)$.*

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Proof. If K_0 is an acumulation point of $e(A)$ in $\mathcal{K}(X)$ then there exists a sequence (K_n) contained in $e(A)$ such that $K_n \overset{H}{\to} K_0$. Thus, $K_n \subseteq A$ for all $n \in \mathbb{N}$ and, due to (1), we have

$$K_0 = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n \geq p} K_n} \subseteq A$$

which implies that $K_0 \in e(A)$.□

Lemma 3 *If A is a nonempty convex subset of X, then $e(A)$ is a nonempty convex subset of $\mathcal{K}(X)$.*

Proof. Let J, K be two nonempty compact subsets of X such that $J, K \in e(A)$. Thus, since $J, K \subseteq A$ and A is a convex set, then

$$\lambda J + (1-\lambda)K = \{\lambda x + (1-\lambda)y : x \in J, y \in K\} \subseteq A,$$

for all $\lambda \in [0, 1]$, which implies that
$\lambda J + (1-\lambda)K \in e(A)$ and, consequently, $e(A)$ is a convex set.□

3 Erraticness and turbulence

In this section we will analyze some connections between the chaos induced by turbulent functions and erratic functions.

Definition 1 *Let $f : X \to X$ be a continuous function. We say that f is a turbulent function if there exist two disjoint nonempty compact subsets J, K of X such that*

$$J \cup K \subseteq f(J) \cap f(K)$$

Definition 2 *Let $f : X \to X$ be a continuous function. We say that f is chaotic in the sense of Block and Coppel (in short: B-C-chaotic) if one of its iterates is turbulent, i.e., there exist $n \geq 1$ and two disjoint nonempty compact subsets J, K of X such that $J \cup K \subseteq f^n(J) \cap f^n(K)$.*

Definition 3 *Suppose that X is a compact convex subset of a linear space and let $f : X \to X$ be a continuous function. We say that f is an erratic function if there exists a nonempty convex compact subset A of X such that*

a) $A \cap f(A) = \emptyset$;

b) $A \cup f(A) \subseteq f^2(A)$.

Theorem 1 *If $f : X \to X$ is a bijective turbulent function, then $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is also a turbulent function.*

Proof. Suppose that f is a turbulent function. Then, if J, K are two disjoint nonempty compact subsets of X such that $J \cup K \subseteq f(J) \cap f(K)$, then by Lemma 1 i)-ii) and Lemma 2 it is clear that $e(J), e(K)$ are two disjoint nonempty compact subsets of $\mathcal{K}(X)$.
Moreover, because $J \cup K \subseteq f(J) \cap f(K)$, we have $J \subseteq f(J) \cap f(K)$ and $K \subseteq f(J) \cap f(K)$, which implies that $e(J) \subseteq e(f(J) \cap f(K))$ and $e(K) \subseteq e(f(J) \cap f(K))$.
Thus, due to equality (2), we obtain

$$\begin{aligned} e(J) \cup e(K) &\subseteq e(f(J) \cap f(K)) \\ &= e(f(J)) \cap e(f(K)) \\ &= \bar{f}(e(J)) \cap \bar{f}(e(K)) \end{aligned}$$

and, consequently, $\bar{f}$ is a turbulent function.□

Corollary 1 *If the continuous bijection f is B-C-chaotic, then $\bar{f}$ is B-C-chaotic.*

Theorem 2 *If $f : X \to X$ is a bijective erratic function, then $\bar{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is also an erratic function.*

Proof. Suppose that f is an erratic function, and let A be a nonempty compact convex subset of X satisfying conditions a) and b) in Definition 3. Then, due to Lemmas 2 and 3, we have that $e(A)$ is a nonempty compact convex subset of $\mathcal{K}(X)$. Now we will prove that $e(A)$ satisfies conditions a) and b) for $\bar{f}$. In fact, by 2 and equality (2) we have:

$$\begin{aligned} i)\ A \cap f(A) = \emptyset &\Rightarrow e(A) \cap e(f(A)) = \emptyset \\ &\Rightarrow e(A) \cap \bar{f}(e(A)) = \emptyset \end{aligned}$$

and, since $A \cup f(A) \subseteq f^2(A)$, we have:

$$\begin{aligned} ii)\ e(A) \cup \bar{f}(e(A)) &= e(A) \cup e(f(A)) \\ &\subseteq e(A \cup f(A)) \\ &\subseteq e(f^2(A)) = \bar{f}^2(e(A)) \end{aligned}$$

which implies that $\bar{f}$ is an erratic function.□

The following result shows that erraticness is stronger than B-C-chaos.

Theorem 3 *Let $f : X \to X$ be a continuous function. If f is an erratic function, then f is a B-C-chaotic function.*

Proof. Suppose that f is an erratic function, and let A be a nonempty compact convex subset of X satisfying conditions a) and b) in Definition 3 Then, due to a), taking $J = A$ and $K = f(A)$ we have $J \cap K = A \cap f(A) = \emptyset$, and by b):

$$\begin{aligned} J \cup K &= A \cup f(A) \\ &\subseteq f^2(A) \\ &= f(K) \\ &\subseteq f(J \cup K) \\ &\subseteq f(f^2(J)) = f^3(J) \end{aligned}$$

and

$$\begin{aligned} J \cup K &= A \cup f(A) \subseteq f^2(J) \\ &\subseteq f^2(J \cup K) \subseteq f^2(f(K)) = f^3(K) \end{aligned}$$

which implies that $J \cup K \subseteq f^3(J) \cap f^3(K)$ and, consequently, f is a B-C-chaotic function.□

Example 1 *If we consider the tent function $f : [0, 1] \to [0, 1]$ defined by*

$$f(x) = \begin{cases} 2x & if \quad 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & if \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

then f is an erratic function. In fact, taking $A = [\frac{1}{4}, \frac{4}{9}]$ we have $f(A) = [\frac{1}{2}, \frac{8}{9}]$, which implies that $A \cap f(A) = \emptyset$ and :

$$A \cup f(A) = \left[\frac{1}{4}, \frac{4}{9}\right] \cup \left[\frac{1}{2}, \frac{8}{9}\right] \subseteq \left[\frac{2}{9}, 1\right] = f^2(A)$$

and, consequently, f is an erratic function.

On the other hand, if we take $J = A$ and $K = f(A)$ then we have $J \cap K = \emptyset$ and, since $f^2(J) = \left[\frac{2}{9}, 1\right]$ and $f^2(K) = [0, 1]$, we obtain that $J \cup K \subseteq f^2(J) \cap f^2(K)$ which implies that f is B-C-chaotic.

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