A Restriction Level Approach for the Representation and Evaluation of Fuzzy Association Rules

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Abstract— *The goal of this paper is to present a logic model with very good properties for the representation and evaluation of fuzzy association rules. For that we are going to combine two approaches. The first one is a model for the representation and evaluation of crisp association rules. The second one is a new proposal for the representation of imprecise properties (in particular for fuzzy sets) by using restriction levels, which verifies all the crisp logic equivalences [1]. Combining both approaches we will achieve a solid model, with very good properties for the representation of fuzzy rules and a very simple framework for extending the crisp quality measures for the evaluation of fuzzy rules.*

Keywords— Logic model, restriction level, RL-probability, RL-numbers, fuzzy association rules.

1 Introduction

Fuzzy association rules have been very developed and have been applied in numerous situations since their appearance in the last nineties. The first works used the Fuzzy Subsets Theory proposed by L.A. Zadeh [2] for putting into groups imprecise values with a clear semantic and later they were used for representing the different types of imprecision found in a stored database. The advantages of using fuzzy sets are on the one hand making smother the bounds and on the other hand give a formal representation for the knowledge semantically significative and meaningful for the user. In [3], [4] we can find a description of some of the most important works in the field of extracting fuzzy association rules.

There are some approaches which offer a framework for generalizing the quality measures for fuzzy association rules (see [5] and references in it). Moreover, although the evaluation measures can be generalized by distinct ways, it should be necessary to search a logic support or an axiomatic base which reflects the choice of a specific measurement. This work tries to follow this line of research and it also tries to establish a logic model that generalizes and preserves in a natural way all the properties of crisp association rules, as well as the different interest measures used in their evaluation. For that we make use of two approaches: the logic model proposed in [6],[7] and the RL-representation theory proposed in [1], [8], [9].

The logic model we are dealing with can manage with different kinds of association rules which might be useful for the user. There are several examples in [10] where the model offers a good formalization for mining exception rules and double rules. Furthermore, we have implemented an algorithm for extracting any kind of rule having the quality measures used for assessing the validity of the rule represented by this model. The algorithm uses the items representation by means of bitsets [11] which reduces de time consuming when computing the conjunction between items and their associated support (bitset cardinality).

The rest of the paper follows with the approach for representing imprecise properties by means of restriction levels presented in [1]. Then we do a brief summary of a model proposed in [7] for the representation of crisp association rules which was widely developed in [6]. The last part of the paper contains our proposal for the representation and the evaluation of fuzzy association rules using a combination of the approaches presented in the previous sections. We finish with the conclusions and possible lines for future research.

2 Representation by Restriction Levels

An imprecise property in an universe X can be represented by a collection of crisp realizations. The approach of [1] for representing imprecise properties by means of restriction levels extends the usual operations from the crisp to the fuzzy case satisfying the logic equivalences between that operations.

Definition 1. [1] A *RL-set* Λ is a finite set of restriction levels $\Lambda = \{\alpha_1, \dots, \alpha_m\}$ verifying that $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$, $m \geq 1$.

In general, the *RL*-set of an atomic property represented by means of a fuzzy set A is defined as follows.

Definition 2. [1] Let be A a fuzzy set defined on the referencial X. Then the *RL*-set associated to A is given by:

$$\Lambda_A = \{A(x) \,|\, x \in X\} \cup \{1\} \tag{1}$$

where $A(x)$ is the grade of membership of x to the fuzzy set A.

The *RL*-set employed for representing an imprecise property is obtained by the union of the *RL*-sets associated to the atomic properties which define that property.

For representing an imprecise property in X by means of restriction levels we are going to use a *RL-representation* defined like a pair (Λ, ρ) where Λ is a *RL*-set and $\rho : \Lambda \to \mathcal{P}(X)$ is a function which applies each restriction level into a crisp realization in this level. For example, the *RL*-representation of an imprecise atomic property defined by a fuzzy set A will be the pair (Λ_A, ρ_A), where Λ_A is given by the equation (1) and $\rho_A(\alpha) = A_\alpha = \{x \in X \,|\, A(x) \geq \alpha\}$ for all $\alpha \in \Lambda_A$.

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Given an imprecise property P represented by (Λ_P, ρ_P), the set of crisp representatives of P, Ω_P is defined as

$$\Omega_P = \{\rho_P(\alpha) \mid \alpha \in \Lambda_P\}. \quad (2)$$

Definition 3. [1] Let be (Λ, ρ) a *RL*-representation with $\Lambda = \{\alpha_1, \ldots, \alpha_m\}$ verifying that $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$. Let $\alpha \in (0,1]$ and $\alpha_i, \alpha_{i+1} \in \Lambda$ satisfying that $\alpha_i > \alpha > \alpha_{i+1}$. Then we define

$$\rho(\alpha) = \rho(\alpha_i). \quad (3)$$

If we look to this definition, this extension for values that there are not in the *RL*-set of the function ρ, is the natural extension if we think in a fuzzy set A and its α-cuts. Using this definition the concept of equivalence between two *RL*-representations is defined.

Definition 4. [1] Let (Λ, ρ) and (Λ', ρ') be two *RL*-representations on X. We will say that both representations are *equivalent* and will be noted by $(\Lambda, \rho) \equiv (\Lambda', \rho')$, if and only if, $\forall \alpha \in (0,1]$

$$\rho(\alpha) = \rho(\alpha'). \quad (4)$$

2.1 Logic Operations

In this section we present a brief overview of the logic operations necessary for the understanding of the rest of the paper. In particular we present the logic operations of disjunction, conjunction and negation that we will need for the generalization of the logic model to the case of fuzzy association rules. The basic ideas of how they are defined can be found in [1].

Definition 5. Let P, Q be two imprecise properties with *RL*-representations (Λ_P, ρ_P), (Λ_Q, ρ_Q). Then, $P \wedge Q$, $P \vee Q$ and $\neg P$ are imprecise properties represented by $(\Lambda_{P\wedge Q}, \rho_{P\wedge Q})$, $(\Lambda_{P\vee Q}, \rho_{P\vee Q})$ and $(\Lambda_{\neg P}, \rho_{\neg P})$ respectively, where

$$\begin{aligned} \Lambda_{P\wedge Q} &= \Lambda_{P\vee Q} = \Lambda_P \cup \Lambda_Q \\ \Lambda_{\neg P} &= \Lambda_P \end{aligned} \quad (5)$$

and, for all $\alpha \in (0,1]$,

$$\begin{aligned} \rho_{P\wedge Q}(\alpha) &= \rho_P(\alpha) \cap \rho_Q(\alpha), \\ \rho_{P\vee Q}(\alpha) &= \rho_P(\alpha) \cup \rho_Q(\alpha), \\ \rho_{\neg P}(\alpha) &= \overline{\rho_P(\alpha)}, \end{aligned} \quad (6)$$

where $\overline{Y}$ is the usual complement of a crisp set Y.

Proposition 1. [1] The operations $\wedge, \vee, \neg$ between *RL*-representations verify the ordinary properties of logic equivalence as for example $\neg\neg A \equiv A$, the Morgan's laws ($\neg(A \wedge B) \equiv (\neg A \vee \neg B)$, one of them) and the law of excluded middle that can be expressed as $A \wedge \neg A \equiv \bot$ or $A \vee \neg A \equiv \top$, where $\top$ and $\bot$ are the atomic properties which represent the tautology (whose *RL*-representation is obtained from the referencial X) and the contradiction (obtained from $\emptyset$).

2.2 RL-numbers

On the basis of *RL*-representations and operations, we introduced in [8] the *RL*-numbers as a representation of fuzzy quantities. This approach offers two main advantages: (1) *RL*-numbers are representations of fuzzy quantities that can be easily obtained by extending usual crisp measurements to fuzzy sets. (2) Arithmetic and logical operations on *RL*-numbers are straightforward and unique extensions of the operations on crisp numbers, verifying the usual properties of crisp arithmetic and logical operations. In addition, the imprecision does not necessarily increase through operations, and it can even diminish. The following definitions and properties are from [8]:

Definition 6. A *RL*-real number is a pair $(\Lambda, \mathcal{R})$ where Λ is a *RL*-set and $\mathcal{R} : (0,1] \to \mathbb{R}$.

We shall note $\mathbb{R}_{RL}$ the set of *RL*-real numbers. The *RL*-real number R_x is the representation of a (precise) real number x iff $\forall \alpha \in \Lambda_{R_x}$, $\mathcal{R}_{R_x}(\alpha) = x$. We shall denote such *RL*-real number as R_x or, equivalently, x, since in the crisp case, the set Λ_{R_x} is not important. Operations are extended as follows:

Definition 7. Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $R_1, \ldots, R_n$ be *RL*-real numbers. Then $f(R_1, \ldots, R_n)$ is a *RL*-real number with

$$\Lambda_{f(R_1,\ldots,R_n)} = \bigcup_{1\le i\le n} \Lambda_{R_i} \quad (7)$$

and, $\forall \alpha \in \Lambda_{f(R_1,\ldots,R_n)}$

$$\mathcal{R}_{f(R_1,\ldots,R_n)}(\alpha) = f(\mathcal{R}_{R_1}(\alpha), \ldots, \mathcal{R}_{R_n}(\alpha)) \quad (8)$$

It is obvious that operations defined in this way are consistent extensions of crisp operations. We want to remark that operations not defined for certain combinations of real values are not defined for *RL*-numbers that verify that combination in at least one restriction level. This is the case of division by 0 for example, i.e., R/R' is defined iff $0 \notin \Omega_{R'}$.

2.3 Probabilities based on RL-numbers

Let us note $[0,1]_{RL} \subseteq \mathbb{R}_{RL}$ the set of *RL*-real numbers verifying $0 \le R \le 1$ $\forall R \in [0,1]_{RL}$ (see definition of *RL*-ranking in [9]). We introduce the notions of *RL*-probability space and *RL*-probability as in [9]:

Definition 8. A *RL*-probability space is a triple (X, Σ, P) where

1. X is a crisp set.
2. Σ is a collection of imprecise events defined by *RL*-representations on X, closed under complement and countable unions, and verifying $X \in \Sigma$.
3. $P : \Sigma \to [0,1]_{RL}$ verifying Kolmogorov's Axioms, i.e., $P(X) = 1$, $P(E) \ge 0$ $\forall E \in \Sigma$, and for any finite collection of disjoint representations $E_1, \ldots, E_n$, $P(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^n P(E_i)$.

In the previous definition, P is a *RL*-probability measure. In particular, *RL*-probability measures can be obtained from ordinary probability measures in an easy way as follows: let (U, F, p) be a (crisp) probability space and let U_F^{RL} be the set of all the *RL*-representations on U such that for any $A \in U_F^{RL}$, $\Omega_A \subseteq F$. By the properties of F, U_F^{RL} is closed under complement and countable unions, and $U \in U_F^{RL}$. For any $A \in U_F^{RL}$, let $P(A) \in [0,1]_{RL}$ defined by $\rho_{P(A)}(\alpha) = p(\rho_A(\alpha)) \forall \alpha \in \Lambda_A$. It is easy to show that (U, U_F^{RL}, P) is a *RL*-probability space and P is a *RL*-probability measure.

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Also easy to show, *RL*-probabilities verify the addition law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and the property $P(\neg A) = 1 - P(A)$. Finally, by definition 8, it is obvious that crisp probabilities are a particular case of *RL*-probabilities since crisp events are particular cases of imprecise events as described by *RL*-representations.

It is easy to check that the usual arithmetic relations between probabilities are preserved in each restriction level and, hence, they are preserved by the arithmetic of *RL*-numbers. For example, $p(A) + p(\neg A) = 1$.

The concept of conditional probability is straightforwardly extended as follows:

Definition 9. Let A and B be two imprecise events defined on U by *RL*-representations $A = (\Lambda_A, \rho_A)$ and $B = (\Lambda_B, \rho_B)$ with $\rho_{p(B)} > 0 \; \forall \alpha \in (0,1]$. The *RL*-probability of A given B in U is

$$p(A|B) = \frac{p(A \wedge B)}{p(B)} \tag{9}$$

Proposition 2. [9] Let A and B be two imprecise events defined on U by *RL*-representations $A = (\Lambda_A, \rho_A)$ and $B = (\Lambda_B, \rho_B)$ with $\rho_{p(B)} > 0 \; \forall \; \alpha \in (0,1]$. Then

$$\rho_{p(A|B)}(\alpha) = \frac{\rho_{p(A \wedge B)}(\alpha)}{\rho_{p(B)}(\alpha)}. \tag{10}$$

Example 1. Let X and Y be atomic imprecise events defined by the following fuzzy sets on $U = \{u_1, \ldots, u_6\}$:

$$\begin{aligned} X &= 1/u_1 + 0.8/u_2 + 0.5/u_3 + 0.4/u_5 \\ Y &= 0.9/u_1 + 0.6/u_3 + 0.5/u_4 \end{aligned} \tag{11}$$

Then we have $\Lambda_X = \Lambda_{\neg X} = \{1, 0.8, 0.5, 0.4\}$ and $\Lambda_Y = \Lambda_{\neg Y} = \{1, 0.9, 0.6, 0.5\}$. The *RL*-set for any operation between X and Y is $\Lambda_X \cup \Lambda_Y$. Table 1 shows the *RL*-representation of X, $\neg X$, Y, $X \wedge Y$, $X \vee Y$ and $X \wedge \neg Y$. Assuming that every u_i is equally probable, table 2 shows the corresponding *RL*-probabilities of imprecise events $p(X)$, $p(\neg X)$, $p(Y)$, $p(X \wedge Y)$, $p(X \vee Y)$ and $p(X \wedge \neg Y)$ and $p(Y|X)$. Remark that the *RL*-probability $p(X|Y)$ is undefined since $\rho_{p(Y)}(1) = 0$.

3 A Logic Model for Association Rules

The logic model we are going to use is based in a method developed in the sixties by Háyek et al. [6]. This method calls GUHA (General Unary Hypotheses Automaton) and it has a good logic and statistical base which contributes to a better understanding of two important aspects of association rules: their nature and the properties of the interest measures used for their evaluation. Recently, several authors have implemented a good and fast algorithm [11] based on a data representation using bit strings. We also use a similar algorithm based on bitsets (an item representation by means of a set of bits) that is low time consuming for computing the items conjunction and their cardinality (support). Having the representation of every item into bitsets computing the contingency table which is based this model is straightforward and low time consuming [11].

Let M be a data matrix where its rows $O_1, \ldots, O_n$ are associated to a set of observed objects, and its columns $A_1, \ldots, A_K$ are the associated attributes which describe the objects. In this way, the entry (i, j) of M will be 1 when the object O_i satisfies the attribute A_j and 0 in other case. In the association rules framework, each matrix M will represent a transaction and the set of all matrixes (transactions) will be the database called D.

D	$< O_1, A_1 >$	$\ldots$	$< O_n, A_K >$
t_1	1	$\cdots$	0
t_2	0	$\cdots$	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$
t_n	1	$\cdots$	1

For the logic model we are presenting, an *item* will be a pair of the form $< O_1, A_1 >$, and an *itemset* will be an aggregation of items using the usual logic connectors: $\wedge, \vee, \neg$. An association rule in the model proposed in [6] is an expression of the type $\varphi \approx \psi$ where φ and ψ represent itemsets (in the sense before) derived from a database D, and the symbol $\approx$ called *quantifier* is an evaluation or condition for the fulfilment of the association rule which will depend on the interest measure used and on the *four fold table*, $4ft$ associated to the itemsets φ and ψ. An example of association rule could be $< O_1, A_1 > \wedge < O_3, A_2 > \approx < O_2, A_5 > \wedge \neg < O_3, A_7 >$.

For any pair of attributes φ and ψ the so called *four fold table* may be constructed from the database D as follows:

$\mathcal{M}$	ψ	$\neg\psi$
φ	a	b
$\neg\varphi$	c	d

This table will be represented by $\mathcal{M} = 4ft(\varphi, \psi, D) = \langle a, b, c, d \rangle$ where a, b, c and d will be non negative integers satisfying that a is the number of objects (i.e. the number of rows of D) which contain at the same time the itemsets φ and ψ, b the number of objects satisfying φ and not ψ, and analogously for c and d. It is obvious that the inequality $a + b + c + d > 0$ is always satisfied.

The association rule $\varphi \approx \psi$ will be true in the database D (or in the matrix $\mathcal{M}$) if and only if the condition associated to the $4ft$-quantifier $\approx$ is satisfied for the four fold table $4ft(\varphi, \psi, D)$.

Depending on the type of $4ft$-quantifier we can express different kinds of associations between the itemsets φ and ψ. In [6] and [7] we can find some examples. The classical framework of support and confidence can be modeled by means of two quantifiers: the *support* and the *implication* $4ft$-quantifiers as follows:

$$\approx_{supp}(a, b, c, d) = \frac{a + b}{a + b + c + d}, \quad \Rightarrow (a, b) = \frac{a}{a + b} \tag{12}$$

which must exceed the $0 < minsupp, minconf < 1$ thresholds respectively imposed by the user. In [12] we explain the existing relation between the $4ft$-quantifiers and the interestingness measures used in the evaluation and validation of association rules.

4 A Logic Model for Fuzzy Association Rules

In [13] the concepts of transaction and association rule are generalized to the fuzzy case.

Definition 10. [13] Let $I = \{i_1, \ldots, i_m\}$ be a finite set of items. A fuzzy transaction is a non empty fuzzy subset $\tilde{\tau} \subseteq I$.

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Table 1: *RL*-representations associated to several imprecise events derived from the atomic properties X and Y.

α_i	$\rho_X(\alpha)$	$\rho_{\neg X}(\alpha)$	$\rho_Y(\alpha)$	$\rho_{X\wedge Y}(\alpha)$	$\rho_{X\vee Y}(\alpha)$	$\rho_{X\wedge\neg Y}(\alpha)$
1	$\{u_1\}$	$\{u_2,u_3,u_4,u_5,u_6\}$	$\emptyset$	$\emptyset$	$\{u_1\}$	$\{u_1\}$
0.9	$\{u_1\}$	$\{u_2,u_3,u_4,u_5,u_6\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1\}$	$\emptyset$
0.8	$\{u_1,u_2\}$	$\{u_3,u_4,u_5,u_6\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1,u_2\}$	$\{u_2\}$
0.6	$\{u_1,u_2\}$	$\{u_3,u_4,u_5,u_6\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_2,u_3\}$	$\{u_2\}$
0.5	$\{u_1,u_2,u_3\}$	$\{u_4,u_5,u_6\}$	$\{u_1,u_3,u_4\}$	$\{u_1,u_3\}$	$\{u_1,u_2,u_3,u_4\}$	$\{u_2\}$
0.4	$\{u_1,u_2,u_3,u_5\}$	$\{u_4,u_6\}$	$\{u_1,u_3,u_4\}$	$\{u_1,u_3\}$	$\{u_1,u_2,u_3,u_4,u_5\}$	$\{u_2,u_5\}$

Table 2: *RL*-probabilities in U of the imprecise events in table 1 and the conditional *RL*-probability $p(Y|X)$.

α_i	$\rho_{p(X)}(\alpha)$	$\rho_{p(\neg X)}(\alpha)$	$\rho_{p(Y)}(\alpha)$	$\rho_{p(X\wedge Y)}(\alpha)$	$\rho_{p(X\vee Y)}(\alpha)$	$\rho_{p(X\wedge\neg Y)}(\alpha)$	$\rho_{p(Y\vert X)}(\alpha)$
1	1/6	5/6	0	0	1/6	1/6	0
0.9	1/6	5/6	1/6	1/6	1/6	0	1
0.8	1/3	2/3	1/6	1/6	1/3	1/6	1/2
0.6	1/3	2/3	1/3	1/6	1/2	1/6	1/2
0.5	1/2	1/2	1/2	1/3	2/3	1/6	2/3
0.4	2/3	1/3	1/2	1/3	5/6	1/3	1/2

For every item $i \in I$ and every transaction $\tilde{\tau}$, an item i will belong to $\tilde{\tau}$ with grade[1] $\tilde{\tau}(i)$ where $\tilde{\tau}(i)$ is a real number in the interval $[0,1]$.

Let $A \subseteq I$ be an itemset. The membership grade of A to the fuzzy transaction $\tilde{\tau}$ is defined as

$$\tilde{\tau}(A) = \min_{i\in A} \tilde{\tau}(i).$$

Using the definition 10 a crisp transaction will be a special case of fuzzy transaction where every item in the transaction will have membership grade equal to 1 or 0 depending on if they are in the transaction or not.

Example 2. We consider the set of items $I = \{i_1,i_2,i_3,i_4,i_5\}$ and the set of fuzzy transactions given by the table 3.

Table 3: Set of fuzzy transactions $\tilde{D}_1$

	i_1	i_2	i_3	i_4	i_5
$\tilde{\tau}_1$	1	0.2	1	0.9	0.9
$\tilde{\tau}_2$	1	1	0.8	0	0
$\tilde{\tau}_3$	0.5	0.1	0.7	0.6	0
$\tilde{\tau}_4$	0.6	0	0	0.5	0.5
$\tilde{\tau}_5$	0.4	0.1	0.6	0	0
$\tilde{\tau}_6$	0	1	0	0	0

In particular, we can see that $\tilde{\tau}_6$ is a crisp transaction. Some membership grade could be: $\tilde{\tau}_1(\{i_3,i_4\})$ =0.9, $\tilde{\tau}_1(\{i_2,i_3,i_4\})$ =0.2 and $\tilde{\tau}_2(\{i_1,i_2\})$ =1.

Definition 11. [13] Let I be a set of itmes, $\tilde{D}$ a set of fuzzy transactions and $A,B \in I$ two disjoint itemsets, i.e. $A\cap B = \emptyset$. A fuzzy association rule is satisfied in $\tilde{D}$ if and only if, $\tilde{\tau}(A) \leq \tilde{\tau}(B)$ for all $\tilde{\tau} \in \tilde{D}$, that is, the membership grade of B is higher than the membership grade of A for all fuzzy transactions $\tilde{\tau}$ in $\tilde{D}$.

This definition maintains the meaning of crisp association rules because if we need that $A \subseteq \tilde{\tau}$ is satisfied, we also need that $B \subseteq \tilde{\tau}$ be satisfied, in our case this can be translated to $\tilde{\tau}(A) \leq \tilde{\tau}(B)$. In this way, as a crisp transaction is a special case of fuzzy transaction, a crisp association rule will be a special case of fuzzy association rule.

Let $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ two fuzzy sets defined on $\tilde{D}$ as $\tilde{\Gamma}_A(\tilde{\tau}) = \tilde{\tau}(A)$ and $\tilde{\Gamma}_B(\tilde{\tau}) = \tilde{\tau}(B)$ respectively. As both are fuzzy sets, we can obtain their own *RL*-representations that we note[2] as $(\Lambda_{\tilde{A}},\rho_{\tilde{A}})$, $(\Lambda_{\tilde{B}},\rho_{\tilde{B}})$, defined as the equations (1) and (3) show.

For the crisp case, some of the suggested measures for the validation of association rules are defined using the concept of probability. This is the case of support and confidence. Using the *RL*-probabilities we can easily extend the classical framework of support and confidence.

Definition 12. (Support of an itemset) Let $A \subseteq I$ be an itemset and $(\Lambda_{p(\tilde{A})},\rho_{p(\tilde{A})})$ the *RL*-representation of the *RL*-probability associated to the fuzzy set $\tilde{\Gamma}_A$ in $\tilde{D}$. Then, the support of A in the fuzzy database $\tilde{D}$ is defined as

$$supp(A) = \sum_{\alpha_i\in\Lambda_{p(\tilde{A})}} (\alpha_i - \alpha_{i+1}) \left(\rho_{p(\tilde{A})}(\alpha_i)\right). \quad (13)$$

Following a similar reasoning we will define the support and the confidence for a fuzzy association rule $A \to B$.

Definition 13. Let be $A,B \subseteq I$ two disjoint itemsets and $(\Lambda_{p(\tilde{A}\wedge\tilde{B})},\rho_{p(\tilde{A}\wedge\tilde{B})})$, $(\Lambda_{p(\tilde{B}|\tilde{A})},\rho_{p(\tilde{B}|\tilde{A})})$ the *RL*-representations of the *RL*-probabilities $p(\tilde{\Gamma}_A \wedge \tilde{\Gamma}_B)$ and $p(\tilde{\Gamma}_B|\tilde{\Gamma}_A)$ in $\tilde{D}$. Then, the support and the confidence of a fuzzy association rule $A \to B$ on $\tilde{D}$ are defined as

$$Supp(A\to B) = \sum_{\alpha_i\in\Lambda_{p(\tilde{A}\wedge\tilde{B})}} (\alpha_i - \alpha_{i+1}) \left(\rho_{p(\tilde{A}\wedge\tilde{B})}(\alpha_i)\right) \quad (14)$$

$$Conf(A\to B) = \sum_{\alpha_i\in\Lambda_{p(\tilde{B}|\tilde{A})}} (\alpha_i - \alpha_{i+1}) \left(\rho_{p(\tilde{B}|\tilde{A})}(\alpha_i)\right) \quad (15)$$

[1]Note that $\tilde{\tau}(i)$ is $\mu_{\tilde{\tau}}(i)$ where $\mu_{\tilde{\tau}} : I \longrightarrow [0,1]$ is the membership function associated to the fuzzy set $\tilde{\tau}$ on the referencial $I = \{$ set of items $\}$.

[2]A and B are crisp sets of disjoint items and $\tilde{\Gamma}_A$, $\tilde{\Gamma}_B$ are fuzzy sets defined on $\tilde{D}$

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We want to remark that in the previous definitions $\Lambda_{p(\tilde{A}\wedge\tilde{B})}$ and $\Lambda_{p(\tilde{B}|\tilde{A})}$ coincides and are equal to $\Lambda_{\tilde{A}} \cup \Lambda_{\tilde{B}}$.

But some other interesting measures are not in terms of probability. For those, the model is necessary for the generalization of those crisp measures of interest.

The following itemsets: $A \wedge B, A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B$ are a partition of the database $\tilde{D}$ (in the same way as $\varphi \wedge \psi, \varphi \wedge \neg\psi, \neg\varphi \wedge \psi$ and $\neg\varphi \wedge \neg\psi$ are a partition for the crisp case). We take their associated fuzzy sets defined on $\tilde{D}$ with their own *RL*-representations that we are going to call $(\Lambda_{\tilde{A}\wedge\tilde{B}}, \rho_{\tilde{A}\wedge\tilde{B}})$, $(\Lambda_{\tilde{A}\wedge\neg\tilde{B}}, \rho_{\tilde{A}\wedge\neg\tilde{B}})$, etc. Note that the *RL*-sets obtained contain the same restriction levels, that is, the set of restriction levels is common and equal to $\Lambda_{\tilde{A}} \cup \Lambda_{\tilde{B}}$.

As for every $\alpha \in \Lambda_Y$, $\rho_Y(\alpha)$ is a crisp set, we can compute its cardinality as usual, and we note it by $|\rho_Y(\alpha)|$.

In this way, for every restriction level $\alpha_i \in \Lambda_{\tilde{A}} \cup \Lambda_{\tilde{B}}$ we define the associated four fold table $\mathcal{M}_{\alpha_i} = 4ft(\tilde{\Gamma}_A, \tilde{\Gamma}_B, \tilde{D}, \alpha_i)$ as follows:

$\mathcal{M}_{\alpha_i}$	$\tilde{\Gamma}_B$	$\neg\tilde{\Gamma}_B$
$\tilde{\Gamma}_A$	a_i	b_i
$\neg\tilde{\Gamma}_A$	c_i	d_i

where a_i, b_i, c_i and d_i are non negative integers such that

$$a_i = |\rho_{\tilde{A}\wedge\tilde{B}}(\alpha_i)|$$
$$b_i = |\rho_{\tilde{A}\wedge\neg\tilde{B}}(\alpha_i)|$$

and analogously with c_i and d_i.

Note that $\forall \alpha_i \in \Lambda_{\tilde{A}} \cup \Lambda_{\tilde{B}}$ the following equality is satisfied

$$a_i + b_i + c_i + d_i = n = |\tilde{D}|. \tag{16}$$

We are going to prove that this definition is a good generalization to the crisp case. Suppose that A and B are two itemsets in a crisp database D, in this case, $\tilde{\Gamma}_A(\tilde{\tau}) = \tilde{\tau}(A) \in [0,1]$ is equivalent to $t(A) \in \{0,1\}$ where $t(A)$ is given by the following indicator function:

$$t(A) = \begin{cases} 1 & \text{si } A \in t \\ 0 & \text{si } A \notin t \end{cases} \tag{17}$$

where t is a crisp transaction of D. The associated *RL*-sets to $\tilde{\Gamma}_A$ and to $\tilde{\Gamma}_B$ will be $\Lambda_{\tilde{A}} = \Lambda_{\tilde{B}} = \{1\}$. And the associated *RL*-representations will be: $(\Lambda_{\tilde{A}}, \rho_{\tilde{A}})$ y $(\Lambda_{\tilde{B}}, \rho_{\tilde{B}})$ where

$$\begin{aligned} \rho_{\tilde{A}}(1) = (\tilde{\Gamma}_A)_1 = \{t \in D | t(A) \geq 1\} = \{t \in D | A \in t\} \\ \rho_{\tilde{B}}(1) = (\tilde{\Gamma}_B)_1 = \{t \in D | t(B) \geq 1\} = \{t \in D | B \in t\} \end{aligned} \tag{18}$$

Following the same process, we can compute the associated *RL*-representations for the sets $\tilde{\Gamma}_A \wedge \tilde{\Gamma}_B$, $\tilde{\Gamma}_A \wedge \neg\tilde{\Gamma}_B$, $\neg\tilde{\Gamma}_A \wedge \tilde{\Gamma}_B$, $\neg\tilde{\Gamma}_A \wedge \neg\tilde{\Gamma}_B$. Therefore, the $4ft$ table for the restriction level $\alpha = 1$ is the same as the $4ft$ table for the itemsets A and B seen in section 3:

$\mathcal{M}_1 \equiv \mathcal{M}$	B	$\neg B$
A	a_1	b_1
$\neg A$	c_1	d_1

Using this representation it is easy to generalize every kind of interest measure used in the crisp case, in particular, we can generalize every $4ft$-quantifier.

If we look again to the classical framework of support and confidence, we can extend it as we show from now on.

Let be $A, B \subseteq I$ two disjoint itemsets and $(\Lambda_{\tilde{A}}, \rho_{\tilde{A}}), (\Lambda_{\tilde{B}}, \rho_{\tilde{B}})$ the *RL*-representations associated to the fuzzy sets $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ in $\tilde{D}$. Then, the support and the confidence of a fuzzy association rule $A \to B$ on $\tilde{D}$ are defined as

$$Supp(A \to B) = \sum_{\alpha_i \in \Lambda_{\tilde{A}}} (\alpha_i - \alpha_{i+1}) \left(\approx_{supp} (a_i, b_i, c_i, d_i)\right), \tag{19}$$

$$Conf(A \to B) = \sum_{\alpha_i \in \Lambda_{\tilde{A}}} (\alpha_i - \alpha_{i+1}) \left(\Rightarrow (a_i, b_i)\right). \tag{20}$$

The support and the confidence defined above coincide with those defined by means of probabilities based on RL-numbers. In this last equation (20) we can find an indetermination of the type "$\frac{0}{0}$", in this case, we will take the value 1 for preserving the definition 11 of fuzzy association rule, because this indetermination happens when there is no transactions satisfying the antecedent and the consequent of the rule among zero transactions satisfying only the antecedent. Again, it is easy to see that for every restriction level α_i the last part of the equations (19) and (20) are the same as the defined support and confidence for a crisp association rule (see quantifiers in (12)).

This process can be done for every measure of interest used in the validation and evaluation of crisp association rules. In particular we use the certainty factor instead of confidence because of its proven good properties: (1) it reduces the number of obtained rules and (2) it provides an useful meaning to the measurement of the validity of association rules. In [12], we prove by means of the logic model that the certainty factor can be seen as an equivalence $4ft$-quantifier as follows:

$$\equiv_{CF} (a,b,c,d) = \begin{cases} \dfrac{ad - bc}{(a+b)(b+d)} & \text{if } ad > bc \\ 0 & \text{if } ad = bc \\ \dfrac{ad - bc}{(a+b)(a+c)} & \text{if } ad < bc. \end{cases} \tag{21}$$

Its generalization for fuzzy rules is straightforward:

$$CF(A \to B) = \sum_{\alpha_i \in \Lambda_{\tilde{A}}} (\alpha_i - \alpha_{i+1}) \left(\equiv_{CF} (a_i, b_i, c_i, d_i)\right). \tag{22}$$

We want to remark that the previous concepts of support, confidence and the certainty factor perfectly extends the support, confidence and certainty factor measures used for crisp association rules. This can be immediately proved because the measures obtained in equations from 19 to 21 coincide with the ones proposed in [13] where it is used a semantic approach based on the evaluation of quantified sentences (see also [14]) using the *GD* method and the fuzzy relative quantifier the majority $Q_M(x) = x$ for evaluating the sentences. In these papers it is proved that using the *GD* method and the quantifier Q_M, the obtained measures in equations (19)-(22) are the ordinary support, confidence and certainty factor for the rule $A \to B$ respectively.

In addition, using the proposed logic model which uses the representation by means of the $4ft$ tables for each restriction

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level, is immediate to prove that if A and B are two itemsets in a set of crisp transactions, the concepts of support, confidence and certainty factor for instance (19)-(22), are equal to the usual crisp concepts of support, confidence and certainty factor (see again equations (12) and (21)).

5 Concluding Remarks

We have presented a simple model with very good properties for the representation and evaluation of fuzzy association rules which extends in a natural way the evaluation of crisp association rules.

We have combined two existing approaches: a logic model for crisp association rules [7] and the recently developed RL-representation theory for imprecise properties [1]. In particular, using the philosophy of RL-representations we define the RL-numbers for the representation of fuzzy quantities [8] and the associated probabilities which are useful for extending crisp quality measures like support and confidence. RL-numbers and probabilities have been also introduced in [15],[16] named gradual numbers. Both approaches are similar with slight differences that mainly affect how is the negation considered [15]. The main similarities and differences between both approaches is treated in [17]. But this is beyond the scope of this paper.

We used RL-numbers and their associated probabilities for the special cases of computing support and confidence of fuzzy association rules but there are many cases that the rule's assessment is made by other measures which are not in terms of probabilities. For these cases we also provide a very simple and systematic approach for extending the crisp measures for mining fuzzy association rules, for instance the certainty factor is extended by equation (22).

In this line of research we have found works like the one in [5]. They develop a systematic approach to the assessment of fuzzy association rules. To this end, they proceed from the idea of partitioning the data stored in a database into examples of the given rules, counterexamples and irrelevant data. Maybe what they called irrelevant data is not so "irrelevant" for measuring the accomplishment of an association rule. See for example the case of the certainty factor in equation (21) which uses the so called irrelevant data (divided into two different frequencies, c and d) for measuring the strength of the rule, and in addition, its good properties are experimentally proved in [13] and formally proved in [12] using the introduced logic model.

Our approach has some similarities with the approach in [5] as we also divide the data, but we use a conjunction-based model, taking all the possible conjunctions between rule's antecedent and consequent. The main difference between the former and our proposal remains in the way of generalizing the quality measures for fuzzy associations, as we use a four fold table's RL-representation.

Since algorithmic aspects of fuzzy rule mining have not been addressed in this paper, let us mention that the method used in [11] for mining crisp rules using the bitset approach can be easily adapted to the fuzzy case and the complexity will increase depending on the number of restrictions levels taken into account.

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