Continuous R-implications

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Abstract— *In this work we have solved an open problem related to the continuity of R-implications. We have fully characterized the class of continuous R-implications obtained from any arbitrary t-norm. Using this result, we also determine the exact intersection between the continuous subsets of R-implications and (S,N)-implications.*

Keywords— R-implication, (S,N)-implication, Łukasiewicz implication, t-norm.

1 Introduction

R-implications and (S,N)-implications are two of the most established families of fuzzy implications. Still, many open problems remain unsolved, see [3, 4]. One of them is related to the continuous subsets of these families. Only recently a characterization of continuous (S,N)-implications was given by the authors in [2].

However, a similar complete characterization regarding the continuous subset of R-implications has not been available so far. It is only known that in the class of R-implications obtained from *left-continuous* t-norms, the only continuous elements are those that are isomorphic to the Łukasiewicz implication, i.e., those R-implications obtained as residuals of nilpotent t-norms. In particular, the following question has remained unanswered so far:

> Does there exist a continuous R-implication obtained from a non-left continuous t-norm?

In this note we show that an R-implication I_T obtained from a t-norm T is continuous if and only if T is a nilpotent t-norm.

Using this result, we are also able to resolve another question related to the intersections between these two families, which is also a generalization of an original result of Smets and Magrez [13], see also [7, 9]. We show that the only continuous (S,N)-implication that is also an R-implication obtained from any t-norm, not necessarily left-continuous, is the Łukasiewicz implication up to an isomorphism.

2 Preliminaries

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

Definition 1 (cf. [7, 10, 9])**.**

(i) A function $N\colon [0,1] \to [0,1]$ is called a *fuzzy negation*, if it is decreasing and satisfies the boundary conditions $N(1) = 0$ and $N(0) = 1$.

(ii) A fuzzy negation N is called *strong*, if it is an involution, i.e., $N \circ N = \mathrm{id}_{[0,1]}$.

(iii) A function $T\colon [0,1]^2 \to [0,1]$ is called a *t-norm*, if it is increasing in both variables, commutative, associative and has 1 as the neutral element.

(iv) A function $S\colon [0,1]^2 \to [0,1]$ is called a *t-conorm*, if it is increasing in both variables, commutative, associative and has 0 as the neutral element.

(v) A t-norm T is said to be *border continuous*, if it is continuous on the boundary of the unit square $[0,1]^2$, i.e., on the set $[0,1]^2 \setminus (0,1)^2$.

(vi) A t-norm T is said to be *left-continuous*, if it is left-continuous in each component.

(vii) A t-norm T is said to be *nilpotent*, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of T, i.e., if there exists $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$, where

$$x_T^{[n]} := \begin{cases} 1, & \text{if } n = 0,\\ x, & \text{if } n = 1,\\ T(x, x_T^{[n-1]}), & \text{if } n > 1. \end{cases}$$

(viii) A t-norm T is said to be *Archimedean* if for every $x, y \in (0,1)$ there exists $n \in \mathbb{N}$ such that $x_T^{[n]} < y$.

Remark 1 (see. [10, p. 17])**.** For the border continuity of a t-norm T, it is sufficient to require the continuity on the upper right boundary, since from the monotonicity we get

$$\lim_{x\to 0^+} T(x,y) \le \lim_{x\to 0^+} T(x,1) = \lim_{x\to 0^+} x = 0 = T(0,y),$$

for any $y \in [0,1]$.

Remark 2. From the commutativity, the left-continuity of a t-norm T is equivalent to the left-continuity of T with respect to the first or the second variable. Moreover, $T(x,1) = 1$ and $T(x,0) = 0$ for every $x \in [0,1]$, thus a t-norm T is left-continuous if and only if for any $y \in (0,1)$ and every increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0,1)$, we have

$$\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y).$$

Proposition 1. *If T is an Archimedean t-norm, then $T(x,y) < \min(x,y)$, for all $x, y \in (0,1)$.*

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Proof. Let T be an Archimedean t-norm. If, on the contrary, there exist some $x_0, y_0 \in (0,1)$ such that $x_0 \geq y_0$ and $T(x_0, y_0) = y_0 = \min(x_0, y_0)$, then we will prove, by induction, that for every $n \in \mathbb{N}$ we have

$$x_{0T}^{[n]} \geq y_0. \tag{1}$$

Indeed, firstly see that

$$x_{0T}^{[0]} = 1 > T(x_0, y_0)) = y_0,$$
$$x_{0T}^{[1]} = x_0 \geq T(x_0, y_0)) = y_0.$$

Let us assume that (1) holds for some $n \in \mathbb{N}$. Then by the monotonicity of T and our inductive assumption we get

$$x_{0T}^{[n+1]} = T(x_0, x_{0T}^{[n]}) \geq T(x_0, y_0) = y_0,$$

which implies that T is not Archimedean, a contradiction. □

By Φ we denote the family of all increasing bijections $\varphi\colon [0,1] \to [0,1]$. We say that two functions $f, g\colon [0,1]^n \to [0,1]$, where $n \in \mathbb{N}$, are Φ-conjugate, if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}\left(f(\varphi(x_1), \ldots, \varphi(x_n))\right),$$

for all $x_1, \ldots, x_n \in [0,1]$. Equivalently, g is said to be the Φ-conjugate of f.

Definition 2 ([7, 4])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

I is decreasing in the first variable, (I1)

I is increasing in the second variable, (I2)

$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0.$ (I3)

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

3 R-implications

Definition 3 (cf. [15, 7, 9])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called an *R-implication*, if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \leq y\}, \tag{2}$$

for all $x, y \in [0,1]$. If an R-implication is generated from a t-norm T, then we will often denote this by I_T.

It is important to note that the name 'R-implication' is a short version of 'residual implication', and I_T is also called as the residuum of T (see e.g. [7, 9, 10]).

Example 1. The Łukasiewicz implication

$$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y), \qquad x, y \in [0,1],$$

is an R-implication obtained from the nilpotent (Łukasiewicz) t-norm

$$T_{\mathbf{LK}}(x,y) = \max(x+y-1, 0), \qquad x, y \in [0,1].$$

For more well-known R-implications along with their t-norms from which they have been obtained, we refer the readers to other sources, notably [7, 10, 4].

Theorem 1 (cf. [7], [3, Theorem 5.5])**.** *If T is any t-norm, then $I_T \in \mathcal{FI}$ and it satisfies the left neutrality property, i.e.,*

$$I_T(1,y) = y, \qquad y \in [0,1], \tag{NP}$$

and the identity principle, i.e.,

$$I_T(x,x) = 1, \qquad x \in [0,1]. \tag{IP}$$

Moreover, if T is left-continuous, then I_T satisfies the exchange principle, i.e.,

$$I_T(x, I_T(y,z)) = I_T(y, I_T(x,z)), \qquad x, y, z \in [0,1],$$

and the ordering property, i.e.,

$$x \leq y \iff I_T(x,y) = 1, \qquad x, y \in [0,1].$$

Proposition 2 ([3, Proposition 5.8])**.** *For a t-norm T the following statements are equivalent:*

(i) T is border continuous.

(ii) I_T satisfies the ordering property.

For R-implications generated form left-continuous t-norms we have the following results.

Proposition 3 (cf. [9, Proposition 5.4.2 and Corollary 5.4.1])**.** *For a t-norm T the following statements are equivalent:*

(i) T is left-continuous.

(ii) T and I_T form an adjoint pair, i.e., they satisfy the residuation property

$$T(x,t) \leq y \iff I_T(x,y) \geq t, \qquad x, y, t \in [0,1].$$

(iii) The supremum in (2) *is the maximum, i.e.,*

$$I_T(x,y) = \max\{t \in [0,1] \mid T(x,t) \leq y\},$$

where the right side exists for all $x, y \in [0,1]$.

Using the above result we are able to obtain the following characterization.

Theorem 2 (cf. [8, Corollary 2])**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is a continuous R-implication based on some left-continuous t-norm.

(ii) I is Φ-conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that

$$I(x,y) = \varphi^{-1}(\min(1 - \varphi(x) + \varphi(y), 1)), \tag{3}$$

for all $x, y \in [0,1]$.

For more facts related to R-implication see e.g. [7, 3, 4].

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4 Continuous Partial Functions of R-implications

Note that from Theorem 1 we can consider, for any fixed $\alpha \in [0,1)$, the decreasing partial function $I_T(\cdot,\alpha)\colon [\alpha,1] \to [\alpha,1]$, which we will denote by g_α^T. Observe that g_α^T is decreasing and such that $g_\alpha^T(\alpha) = 1$ and $g_\alpha^T(1) = \alpha$.

Remark 3. If the domain of g_α^T is extended to $[0,1]$, then this is exactly what are called contour lines by Maes and De Baets in [11, 5]. If $\alpha = 0$, then g_0^T is the natural negation associated with the t-norm T (see [3]).

Theorem 3. *Let T be any t-norm and $\alpha \in [0,1)$. If g_α^T is continuous, then g_α^T is strictly decreasing.*

Proof. Let T be any t-norm and $\alpha \in [0,1)$ be fixed. We know that g_α^T is decreasing. On the contrary, let us assume that g_α^T is constant on some interval $[x_0,y_0]$ for some $\alpha < x_0 < y_0 < 1$, i.e., there exists $p \in [\alpha,1]$ such that

$$g_\alpha^T(x_0) = g_\alpha^T(y_0) = p.$$

Let us fix arbitrarily $z \in (x_0,y_0)$.

Firstly, consider the case $p = 1$. Then

$$g_\alpha^T(z) = I_T(z,\alpha) = \sup\{t \in [0,1] \mid T(z,t) \le \alpha\} = 1,$$

thus $T(z,1-\varepsilon) \le \alpha$ for any $\varepsilon \in (0,1)$. Hence

$$g_\alpha^T(1-\varepsilon) = \sup\{t \in [0,1] \mid T(1-\varepsilon,t) \le \alpha\} \ge z,$$

for all $\varepsilon \in (0,1-\alpha)$. However, by the continuity of g_α^T, as $\varepsilon \to 0^+$, we get

$$\begin{aligned}\alpha = g_\alpha^T(1) &= g_\alpha^T(\lim_{\varepsilon\to 0^+} 1-\varepsilon) = \lim_{\varepsilon\to 0^+} g_\alpha^T(1-\varepsilon)\\ &\ge \lim_{\varepsilon\to 0^+} z = z,\end{aligned}$$

a contradiction to the fact that $\alpha < x_0 < z$.

If $p = \alpha$, then

$$g_\alpha^T(z) = I_T(z,\alpha) = \sup\{t \in [0,1] \mid T(z,t) \le \alpha\} = \alpha,$$

thus $T(z,\alpha+\varepsilon) > \alpha$ for all $\varepsilon \in (0,1-\alpha)$. Hence

$$g_\alpha^T(\alpha+\varepsilon) = \sup\{t \in [0,1] \mid T(\alpha+\varepsilon,t) \le \alpha\} \le z,$$

for all $\varepsilon \in (0,1-\alpha)$. Once again, by the continuity of g_α^T we have, as $\varepsilon \to 0^+$, that

$$\begin{aligned}1 = g_\alpha^T(\alpha) &= g_\alpha^T(\lim_{\varepsilon\to 0^+} \alpha+\varepsilon) = \lim_{\varepsilon\to 0^+} g_\alpha^T(\alpha+\varepsilon)\\ &\le \lim_{\varepsilon\to 0^+} z = z,\end{aligned}$$

a contradiction to the fact that $z < 1$.

Finally, let $p \in (\alpha,1)$. Then, by the definition of g_α^T, we have

$$T(z,p+\varepsilon) > \alpha \ge T(z,p-\varepsilon),$$

for any $\varepsilon > 0$ such that $p+\varepsilon \le 1$ and $p-\varepsilon \ge \alpha$. Therefore

$$I_T(p+\varepsilon,\alpha) \le z \le I_T(p-\varepsilon,\alpha),$$

hence

$$g_\alpha^T(p+\varepsilon) \le z \le g_\alpha^T(p-\varepsilon).$$

Since g_α^T is continuous, we have, as $\varepsilon \to 0^+$, that

$$g_\alpha^T(p) = z.$$

Now this happens for every $z \in (x,y)$, which contradicts the fact that g_α^T is a function itself. Hence g_α^T is strictly decreasing. □

5 Main results: continuous R-implications

The main result of this work is the generalization of Theorem 2, viz., we show that the *left-continuity* of the underlying t-norm is implied and need not be assumed. Thus we give a complete characterization of the class of all continuous R-implications by showing that it is equivalent to the class of fuzzy implications which are Φ-conjugate to the Łukasiewicz implication.

Theorem 4. *Let T be a t-norm and I_T the R-implication obtained from it. If I_T is continuous, then T is border continuous.*

Proof. On the contrary, let us assume that I_T is continuous and T is not border continuous. Then, by Remark 1, there exist $y_0 \in (0,1)$ and an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0,1)$, such that $\lim_{n\to\infty} x_n = 1$, but

$$\lim_{n\to\infty} T(x_n,y_0) = y' < y_0 = T(1,y_0).$$

This implies, in particular, that

$$I_T(y_0,y') = \sup\{t \in [0,1] \mid T(y_0,t) \le y'\} = 1.$$

Now, by (I1) and (IP) of I_T (cf. Theorem 1) we have that

$$1 = I_T(y_0,y') \le I_T(y',y') = 1,$$

i.e., $I_T(x,y') = 1$ for all $x \in [y',y_0]$. Note that $I_T(\cdot,y') = g_{y'}^T$. Since I_T is continuous we have that $g_{y'}^T$ is also continuous and from Theorem 3 we see that it is strictly decreasing. However, from the above, we see that $g_{y'}^T$ is constant on $[y',y_0]$, a contradiction. Thus T is border continuous. □

Theorem 5. *Let T be a t-norm and I_T the R-implication obtained from it. If I_T is continuous, then T is Archimedean.*

Proof. Let T be a t-norm. On the contrary, let us assume that I_T is continuous and T is non-Archimedean. Then, by the Definition 1(viii) there exist $x_0, y_0 \in (0,1)$ such that for all $n \in \mathbb{N}$ we have that ${x_0}_T^{[n]} \ge y_0$.

Let us denote

$$X_0 := \{z \in [0,1] \mid {x_0}_T^{[n]} > z \text{ for all } n \in \mathbb{N}\}.$$

Observe, that $X_0 \neq \emptyset$ since for all $y < y_0$ we have that ${x_0}_T^{[n]} > y$ for all $n \in \mathbb{N}$. Further, let

$$z_0 := \sup X_0.$$

See that $0 < z_0 \le x_0$ and $z_0 - \varepsilon \in X_0$ for all $\varepsilon \in (0,z_0]$. Also, if $t > z_0$, then there exists $m \in \mathbb{N}$ such that ${x_0}_T^{[m]} \le t$, which implies that

$$z_0 - \varepsilon < {x_0}_T^{[m+1]} = T(x_0, {x_0}_T^{[m]}) \le T(x_0,t),$$

for any $t > z_0$. Hence

$$I_T(x_0,z_0-\varepsilon) = \sup\{t \in [0,1] \mid T(x_0,t) \le z_0-\varepsilon\} \le z_0,$$

for all $\varepsilon \in (0,z_0]$. From the continuity of I_T we get

$$\begin{aligned}I_T(x_0,z_0) &= I_T(x_0, \lim_{\varepsilon\to 0^+} z_0-\varepsilon) = \lim_{\varepsilon\to 0^+} I_T(x_0,z_0-\varepsilon)\\ &\le \lim_{\varepsilon\to 0^+} z_0 = z_0.\end{aligned}$$

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Now, by (I1) and (NP) of I_T (cf. Theorem 1) we have that

$$z_0 \geq I_T(x_0, z_0) \geq I_T(1, z_0) = z_0,$$

i.e., $I_T(x, z_0) = z_0$ for all $x \in [x_0, 1]$. Note that $I_T(\cdot, z_0) = g^T_{z_0}$. Since I_T is continuous we have that $g^T_{z_0}$ is also continuous and from Theorem 3 we see that it is strictly decreasing. However, from the above, we have that $g^T_{z_0}$ is constant on $[x_0, 1]$, a contradiction. Thus T is Archimedean. □

Theorem 6. *Let T be a t-norm and I_T the R-implication obtained from it. If I_T is continuous, then T is left-continuous.*

Proof. Let T be a t-norm such that I_T is continuous. From Theorems 4 and 5 we see that T is border continuous and Archimedean.

On the contrary, let us assume that T is non-left-continuous. From Remark 2 there exist $x_0 \in (0, 1]$, $y_0 \in (0, 1)$ and an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0, 1)$, such that $\lim_{n\to\infty} x_n = x_0$, but

$$\lim_{n\to\infty} T(x_n, y_0) = z' < z_0 = T(x_0, y_0).$$

Since T is border continuous it suffices to consider the case when $x_0 \in (0, 1)$.

Firstly observe that

$$I_T(y_0, z') = \sup\{t \in [0, 1] \mid T(y_0, t) \leq z'\} = x_0, \qquad (4)$$

since from the monotonicity of T we have $T(y_0, x_n) \leq z'$ for every $n \in \mathbb{N}$ and $T(y_0, x_0) = z_0 > z'$.

Next, from Proposition 1, by the Archimedeanness and monotonicity of T, we get that for any arbitrary $\varepsilon \in (0, 1-x_0)$ the following inequality holds

$$T(x_0, 1 - \varepsilon) < \min(x_0, 1 - \varepsilon) = x_0. \qquad (5)$$

Now, by (4) and (5) we get

$$T(x_0, 1 - \varepsilon) < I_T(y_0, z'),$$

for any $\varepsilon \in (0, 1 - x_0)$, thus

$$T(x_0, 1 - \varepsilon) < \sup\{t \in [0, 1] \mid T(y_0, t) \leq z'\},$$

hence

$$T(y_0, T(x_0, 1 - \varepsilon)) \leq z'$$

By the associativity of T we get

$$T(T(x_0, y_0), 1 - \varepsilon) \leq z',$$

i.e.,

$$T(z_0, 1 - \varepsilon) \leq z'.$$

for any $\varepsilon \in (0, 1 - x_0)$. This implies that

$$\lim_{\varepsilon\to 0^+} T(z_0, 1 - \varepsilon) \leq z' < z_0 = T(z_0, 1),$$

i.e., T is not border continuous, a contradiction to Theorem 4 and hence T is left-continuous. □

From Theorems 2 and 6 we obtain the following result.

Corollary 1. *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) I is a continuous R-implication based on some t-norm.

(ii) I is Φ-conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that I has the form (3) *for all $x, y \in [0, 1]$.*

Let us denote the different families of fuzzy implications as follows:

- $\mathbb{I}_{\mathbb{T}}$ – the family of all R-implications;
- $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ – the family of all R-implications obtained from left-continuous t-norms;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{T}}$ – the family of all continuous R-implications;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ – the family of all continuous R-implications obtained from left-continuous t-norms;
- $\mathbb{I}_{\mathbb{LK}}$ – the family of all fuzzy implications Φ-conjugate with the Łukasiewicz implication $I_{\mathbf{LK}}$.

From Corollary 1 we have the following equalities between the above sets:

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = {}^{\mathbb{C}}\mathbb{I}_{\mathbb{T}} = \mathbb{I}_{\mathbb{LK}}.$$

6 (S,N)-implications

Definition 4 (cf. [15, 7, 1, 2])**.** A function $I: [0, 1]^2 \to [0, 1]$ is called an *(S,N)-implication*, if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \qquad x, y \in [0, 1].$$

If N is a strong negation, then I is called an S-implication. Moreover, if an (S,N)-implication is generated from S and N, then we will often denote this by $I_{S,N}$.

Firstly note that $I_{S,N} \in \mathcal{FI}$ for any t-conorm S and any fuzzy negation N. In the class of continuous (S,N)-implications we have the following important result.

Proposition 4 ([2, Proposition 5.4])**.** *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) I is a continuous (S,N)-implication.

(ii) I is an (S,N)-implication generated from some continuous t-conorm S and some continuous fuzzy negation N.

Let us denote the different families of fuzzy implications as follows:

- $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ – the family of all (S,N)-implications;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}}$ – the family of all continuous (S,N)-implications;
- $\mathbb{I}_{\mathbb{S}_{\mathbb{C}},\mathbb{N}_{\mathbb{C}}}$ – the family of all (S,N)-implications obtained from continuous t-conorms and continuous negations.

Observe, that from Proposition 4 we get

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}} = \mathbb{I}_{\mathbb{S}_{\mathbb{C}},\mathbb{N}_{\mathbb{C}}}.$$

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7 Intersection between continuous R- and (S,N)-implications

The intersections between the families and subfamilies of R- and (S,N)-implications have been studied by many authors, see e.g. [6, 13, 7, 3]. As regards the intersection between their continuous subsets only the following result has been known so far.

Theorem 7. *The only continuous (S,N)-implications that are also R-implications obtained from left-continuous t-norms are the fuzzy implications which are* Φ*-conjugate with the Łukasiewicz implication, i.e.,*

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{LK}}.$$

Now, from Corollary 1, we can prove the following equivalences.

Theorem 8. *For a function* $I\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(i) I is a continuous (S,N)-implication that is also an R-implication obtained from a left-continuous t-norm.

(ii) I is a continuous (S,N)-implication that is also an R-implication.

(iii) I is an (S,N)-implication that is also a continuous R-implication.

(iv) I is Φ*-conjugate with the Łukasiewicz implication, i.e., there exists* $\varphi \in \Phi$*, which is uniquely determined, such that I has the form* (3) *for all* $x, y \in [0,1]$*.*

In summary, we have

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = {}^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}} = \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap {}^{\mathbb{C}}\mathbb{I}_{\mathbb{T}} = {}^{\mathbb{C}}\mathbb{I}_{\mathbb{T}} = \mathbb{I}_{\mathbb{LK}}.$$

8 Conclusions

In this paper, we have shown that continuous R-implications cannot be obtained from purely non-left-continuous t-norms and that the only continuous R-implications are those that are Φ-conjugate with the Łukasiewicz implication. Using this result we have been able to answer another question related to the intersection between the continuous sub-families of (S,N)- and R-implications. We believe that this work will further help in solving some of the open problems still remaining with regards these two basic families of fuzzy implications.

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